

On a conjecture of Ayala–Francis–Rozenblyum

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(work done in part in Zurich)

1. The smooth realisation problem
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The smooth realisation problem

X : finite homotopy type

Question:

$$X \simeq M ?$$

for smooth M ?

Answer:

- **Yes** for M open, or compact with boundary
 - embed $X \hookrightarrow \mathbb{R}^N$ and thicken to regular nbhd
- **No** for M closed (*geometry/topology since the 50s*)
 - Poincaré duality, finitely presented π_1 (conversely $K(\pi, 1) \simeq M^4$ ✓)
 - $n \geq 5 \Rightarrow$ Spivak normal fib + surgery obstruction in $L_n(\mathbb{Z}[\pi_1])$
 - $n = 4 \Rightarrow$ surgery harder; Donaldson, Seiberg–Witten; open
 - $n = 3 \Rightarrow$ prime decomposition [Kneser, Milnor] + geometrisation [Thurston, Perelman]; primes: $S^1 \times S^2$, S^3/Γ , universal cover aspherical $\xrightarrow{\text{GC}} K(\pi, 1)$ with PD group π
 - ...

The stratified generalisation

$$\begin{aligned} \infty\text{-groupoids} &\rightsquigarrow \infty\text{-categories} \\ \text{Sing}(M) &\rightsquigarrow \text{Exit}(M) \subset \text{Sing}(M) \end{aligned}$$

Generalisation:

$$\boxed{\mathcal{C} \simeq \text{Exit}(M) ?}$$

Get $\text{Exit}(X) \subset \text{Sing}(M)$ by *excluding paths*

using a good **stratification**

$$s: M \rightarrow P_{\text{Aleksandrov}}$$

- $\mathcal{C}$: directed (*layered*) + appropriately *finite*
- (M, s) : appropriately *smooth*

Recall:

$$\mathrm{Sing}_n(M) = \{\Delta^n \rightarrow M\}$$

where

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_0 + \dots + t_n = 1\}$$

- Sing_0 : points $x \in M$ are objects,
- Sing_1 : paths $x = \gamma(0) \xrightarrow{\gamma} \gamma(1) = y$ are 1-morphisms
- Sing_2 : homotopies $\gamma: \Delta^2 \rightarrow M$:

$$\begin{array}{ccc} & y & \\ d_2\gamma \nearrow & & \searrow d_0\gamma \\ x & \xrightarrow{d_1\gamma} & z \end{array}$$

are 2-morphisms ' $d_2\gamma \circ d_0\gamma \Rightarrow d_1\gamma$ ', etc.

∞ -category: can compose but not (necessarily) invert 1-morphisms

like groups $\rightsquigarrow$ monoids

$\text{Exit}(M) \subset \text{Sing}(M)$ by *excluding paths* via $s: M \rightarrow P_{\text{Aleks}}$

e.g.

$$s: \mathbb{R}^2 \rightarrow \{0 < 1\},$$

$$0 \mapsto 0$$

$$0 \neq z \mapsto 1$$

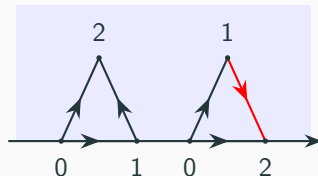
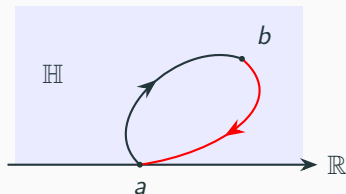
Conical singularities:

$$C(Z) = 0 \amalg_{0 \times Z} Z \times [0, \infty)$$

above: $(\mathbb{R}^2, s) \cong C(S^1)$

Only allow simplices lying over P !

Want: $\gamma: \Delta^n \rightarrow M$ such that $s(\gamma(t_0, \dots, t_i, 0, \dots, 0)) = p_i$ for an ascending sequence $p_0 \leq \dots \leq p_n$



[Treumann, Lurie]

Strict pullback:

$$\begin{array}{ccc} \mathrm{Exit}(M) & \longrightarrow & \mathrm{Sing}(M) \\ \downarrow & & \downarrow \\ N(P) & \longrightarrow & \mathrm{Sing}(P_{\mathrm{Aleks}}) \end{array}$$

where $(p_0 \leq \dots \leq p_n) \mapsto (\sigma: (t_0, \dots, t_i, 0, \dots, 0) \mapsto p_i)$.

Not always an ∞ -category!

$\mathrm{Exit}(M) \rightarrow N(P)$ should be nice.

Conical [Lurie–MacPherson], **conically smooth** [Ayala–Francis–Tanaka],
Whitney (e.g. analytic varieties) [Nocera–Volpe] ✓

A naive approach

$$\mathcal{C} \rightsquigarrow |\mathcal{C}|$$

Is there $s: |\mathcal{C}| \rightarrow P$ such that $\mathcal{C} \simeq \text{Exit}(|\mathcal{C}|)$?

$|-|$ is homotopically out of control: adds topology from ‘algebra’

- $M = \{1, x_{ij} : i, j = 1, 2, x_{ij}x_{kl} = x_{il}\} \Rightarrow BM \simeq S^2$ [Fiedorowicz]
- $B\text{Homeo}(M)^\delta \xrightarrow{\sim} B\text{Homeo}(M)$ in homology [Thurston]
- X connected CW complex $\Rightarrow X \simeq BG^\delta$ in homology [Kan–Thurston]
- X connected CW complex $\Rightarrow X \simeq BM^\delta$ [McDuff]
- Without non-trivial endomorphisms: $|* \xrightarrow{S^1} *| \simeq \Sigma S^1$

Directedness + finiteness

Directedness: conservative functor $s: \mathcal{C} \rightarrow P$ (P -layered ∞ -cat)

Every endo is an iso

$$P = [1] = \{0 < 1\}$$

Strata: $\mathcal{C}_0, \mathcal{C}_1 \leftarrow \mathbf{finite}$

Global paths: $(\mathcal{C}_0 \downarrow \mathcal{C}_1) = \mathcal{C}_0 \times_{\mathcal{C}\{0\}} \mathcal{C}^{\Delta^1} \times_{\mathcal{C}\{1\}} \mathcal{C}_1$: the *link* $\leftarrow \mathbf{finite}$

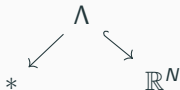
Local paths: $(x \downarrow \mathcal{C}_1)$: *local links* $\leftarrow \mathbf{finite}$ (without this: [Volpe]!)

$$\begin{array}{ccc} & (\mathcal{C}_0 \downarrow \mathcal{C}_1) & \\ \sigma \swarrow & & \searrow \tau \\ \mathcal{C}_0 & & \mathcal{C}_1 \end{array}$$

The conjecture and the counterexamples

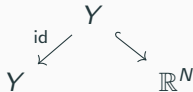
Conjecture (Ayala–Francis–Rozenblyum '15). *Every finite ∞ -category $\mathcal{C}$ in which every endomorphism is an isomorphism satisfies $\mathcal{C} \simeq \text{Exit}(M)$ for a conically smooth manifold M .*

Class I. Will construct finite ∞ -categories $\mathcal{E}$ of type



s.t. $\mathcal{E} \not\simeq \text{Exit}(M)$, with Λ closed + Stiefel–Whitney $w \neq 1 \in H^*(M; \mathbb{Z}_2)$.

Class II. Will construct finite ∞ -categories $\mathcal{E}$ of type



such that $\mathcal{E} \not\simeq \text{Exit}(M)$ for any compact M , with Y closed + non-homology-sphere.

Conical smoothness

Locally: $\mathbb{R}^n \times C(Z)$ with compact Z

Globally: Spherical blow-up:

$$\begin{array}{ccc} \text{Link}(M) & \xrightarrow{\text{boundary}} & \text{Unzip}(M) \\ \downarrow & & \downarrow \\ M \setminus M^\circ = S_M & \hookrightarrow & M \end{array}$$

The cone:

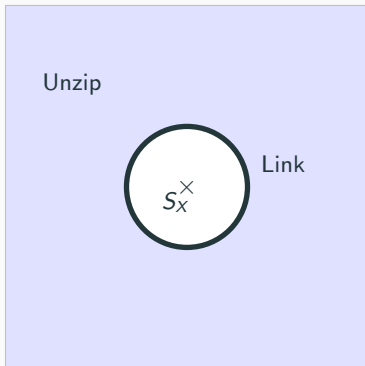
$$\begin{array}{ccc} \{0\} \times Z & \hookrightarrow & [0, \infty) \times Z \\ \downarrow & & \downarrow \\ * & \xrightarrow{\text{cone pt}} & C(Z) \end{array}$$

'Polar coordinates'

$$\mathbb{R}^2 \rightarrow \{0 < 1\},$$

$$0 \mapsto 0$$

$$0 \neq z \mapsto 1$$

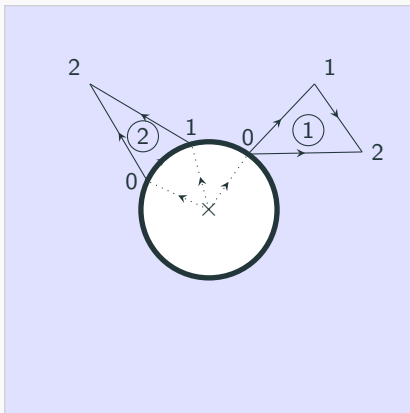


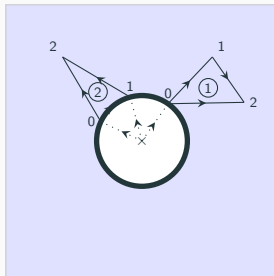
The main theorems

$$\mathfrak{S} = \left(\begin{array}{ccc} & L & \\ \pi \swarrow & & \searrow \iota \\ M & & N \end{array} \right)$$

Want: Concrete construction of an ∞ -category of type $\mathfrak{S}$.

Idea: adjoin non-invertibles formally:





$$\text{EX}(\mathfrak{S}) = M + N + \text{non-invertibles}$$

$$(\Gamma, j) \in N_k \times \{1, \dots, k\}, \quad \Gamma|_{0, \dots, j-1} \in \iota(L)$$

$$d_i(\Gamma, j) = (d_i \Gamma, b_{j,i})$$

$$s_i(\Gamma, j) = (s_i \Gamma, \sharp_{j,i})$$

$$b_{j,i} = \begin{cases} j, & i \geq j \\ j-1, & i < j \end{cases} \quad \sharp_{j,i} = \begin{cases} j, & i \geq j \\ j+1, & i < j \end{cases}$$

Theorem (T '25) $\mathrm{EX}(M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ is an ∞ -category whenever M, L, N are ∞ -categories, π is a right fibration, and ι is a cofibration.

(Linked ∞ -categories/spaces.)

Theorem (T '25) $\mathrm{EX}: \mathrm{LS} \hookrightarrow \mathrm{Cat}_{\infty}$ is a fully faithful ∞ -functor.

Theorem (T '25, Ivanov–T '26) The image of EX includes $\mathrm{Exit}(M)$ for conically smooth M .

Over $P = \{0 < 1\}$: $\mathrm{Exit}(M) \simeq \mathrm{EX}(M_0 \leftarrow \mathrm{Link} \hookrightarrow \mathrm{Unzip}^{\circ} \cong M_1)$

The proofs of the counterexamples

$\mathfrak{G} = (* \leftarrow \Lambda \hookrightarrow \mathbb{R}^N)$ with Λ closed and with $w(\Lambda) \neq 0$

E.g. $\mathbb{R}P^n$ with $n+1 \neq 2^k$ since

$$w(\mathbb{R}P^n) = (1 + \alpha)^{n+1} \in H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$$

$$\text{Exit}(M) \simeq \text{EX}(\mathfrak{G}) \Rightarrow$$

$$\left(\begin{array}{ccc} & \Lambda & \\ \swarrow & & \searrow \\ * & & \mathbb{R}^N \end{array} \right) \simeq \left(\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ M_0 & & M_1 \end{array} \right)$$

M_1 framed, $L \hookrightarrow M_1$ has normal framing $\Rightarrow L$ stably framed

$L \rightarrow M_0 \simeq *$ is a smooth Z -bundle $\Rightarrow Z$ stably framed

but $\Rightarrow Z \simeq Z \times M_0 \simeq L \simeq \Lambda \Rightarrow w(\Lambda) = 0!$

since both Z and Λ are compact

$\mathfrak{S} = (Y \leftarrow Y \hookrightarrow \mathbb{R}^N)$ with Y closed and not a homology sphere
 ($Y = S^1 \times S^1$)

$\text{Exit}(M) \simeq \text{EX}(\mathfrak{S})$ for *compact* $M \Rightarrow$

$$\left(\begin{array}{ccc} & Y & \\ Y \swarrow & & \searrow \\ & \mathbb{R}^N & \end{array} \right) \simeq \left(\begin{array}{ccc} & L & \\ M_0 \swarrow & & \searrow \\ & M_1 & \end{array} \right)$$

Consider $x \in M_0$, $x \in U \cong \mathbb{R}^k \times C(Z) \subset M$.

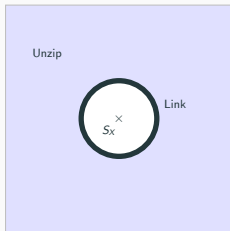
$(L \rightarrow M_0) \simeq (\text{id}: Y \rightarrow Y)$ gives $Z = *$ **because Z is compact**

But then $U \cong \mathbb{R}^k \times C(*) = \mathbb{H}^{k+1}$, and M is smooth with boundary.

$M_1 \simeq * \Rightarrow M \simeq * \Rightarrow \partial M = M_0 \simeq Y$ is a homology k -sphere

(Lefschetz duality, using that M is **compact**; non-balls in 4d: Mazur!)

**Outlook: ∞ -categories via
geometric topology?**



$$\begin{array}{ccc}
 \text{Link}(M) & \xrightarrow{\text{boundary}} & \text{Unzip}(M) \\
 \downarrow & & \downarrow \\
 M \setminus M^\circ = S_M & \hookrightarrow & M
 \end{array}$$

The existence of M is a condition on $M^\circ = M_1!$

$M^\circ \cong \text{Unzip}(M)^\circ$ is the interior of a smooth manifold with **corners**.

M° collapses into S_M smoothly at infinity (but not everywhere)

Lenght 1 $\Rightarrow$ Unzip(M) is smooth with boundary: classical!

Question: When is open M_1 the interior of a **compact** smooth manifold $M = \overline{M_1}$ with boundary?

Answer: [Siebemann, $n = \dim(M) \geq 6$ because s-cobordism] finitely many tame ends ϵ_i + the *Siebenmann obstruction*(s) $\sigma(\epsilon_i) \stackrel{!}{=} 0 \in \tilde{K}_0(\mathbb{Z}\pi_1^\infty(\epsilon_i))$ must vanish.

Some remarks:

- This is an end-wise obstruction; we don't want to bound all ends ✓
- Holds topologically for $n \geq 5$ (Freedman's s-cob); open for $n = 4...$
- $\overline{\partial M_1} \stackrel{!}{=} \text{Link}$ **is unique up to diffeomorphism!** ($n \geq 6!$)
- Our case: $\overline{\partial M_1} \rightarrow S_M = M_0$ smooth Z -bundle for a *closed* $Z \simeq$ the local link $\Rightarrow$ restrictions on Link from the Serre spectral sequence

No obstruction without the compactness requirement.