Talk 11: The AKSZ construction in the derived setting and relation to (semi-)classical TFTs

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Except for a few remarks, everything can be found in [Cal14]; [Cal15]; [Pan+13]. The fully extended version seems to be still in the works [CHS].

Contents

1 Recollections

1.1 Stacks

Throughout, *stacks* are derived ∞-stacks

$$\mathsf{dAff}^{\mathrm{op}} = \mathsf{cdga}_{\leq 0} \to \mathsf{sSet}$$

that preserve weak equivalences, land in weak Kan complexes ($\sim$ weak ∞-groupoids), and satisfy étale descent. We denote by dSt the $(\infty, 1)$-category (Lurie's ∞) of stacks.

Any derived 1-stack (with target Set, landing in groupoids) gives a stack after taking nerves.

Similar to sheafification, *stackification* associates a stack to a simplicial presheaf on affine derived schemes, which amounts to taking fibrant replacements in a natural model structure on simplicial preasheaves (induced by one on sSet where weak Kan complexes are fibrant).

Definition 1.1 (Betti stack)**.** Let X be a topological space and let $\mathrm{Sing}_\bullet(X)$ denote its singular chains. The stackification of

$$X_B \colon A \mapsto \mathrm{Sing}_\bullet(X),$$

still denoted by X_B, is called the *Betti stack* associated to X.

Definition 1.2 (Mapping stack)**.** Given stacks X, Y, we set

$$\mathbf{Map}(X,Y)(A) = \mathrm{Hom}_{\mathsf{dSt}}(X \times A, Y).$$

This defines a stack $\mathbf{Map}(X,Y)$.

Where necessary, we implicitly treat affine derived schemes as stacks.

1.2 Symplectic structures

Let $A \in \mathsf{cdga}_{\leq 0}$ and let Ω^1_A denote the A-module of its Kähler differentials (1-forms). Let

$$\Omega^p_A = \mathrm{Sym}^p(\Omega^1_A[1])[-p].$$

Choose a fibrant replacement $\widetilde{A} \to A$ (under the base field $\Bbbk$) with respect to the projective model structure on $\mathsf{cdga}_{\leq 0}$, and write

$$\mathcal{A}^p(A) = \Omega^p_{\widetilde{A}} \in \mathsf{Cpx},$$

considered as a $\Bbbk$-complex. For any stack X, we may set $\mathcal{A}^p(X) = \mathrm{holim}_{\mathrm{Spec}(A)\to X} \mathcal{A}^p(A)$. Finally, write

$$\mathcal{A}^p(X,n) = \mathrm{Hom}_{\mathsf{Cpx}}(\Bbbk, \mathcal{A}^p(X)[n])$$

for the *space* of n-cocycles in $\mathcal{A}^p(X)$, called *p-forms of degree n*.

Definition 1.3.

- The *de Rham algebra* of A,

  $$\mathrm{DR}^\bullet(A) = \prod_{n\geq 0} \Omega^n_A[-n]$$

  has total differential $d_\Omega + d_{\mathrm{dR}}$, with d_Ω the internal differential of each $\Omega^n_A[-n]$, and d_{dR} is the de Rham differential extending $A \to \Omega^1_A[-1]$, where it is the usual de Rham differential with a shift.

- Let

  $$\mathcal{A}^{p,\mathrm{cl}}(A) = \mathrm{Hom}_{\mathsf{Cpx}}(\Bbbk, \mathrm{DR}^{\geq p}(\widetilde{A})[p][n])$$

  denote the space of *closed p-forms of degree n*.

- Let the A-module

  $$\mathbb{L}_A = A \otimes_{\widetilde{A}} \Omega^1_{\widetilde{A}}$$

  denote the *cotangent complex* of A, which again extends to any stack.

As $\mathcal{A}^{p,\mathrm{cl}}$ satisfies étale descent again, it is defined on any stack, just like $\mathcal{A}^p$.

Definition 1.4. Let X be an Artin stack locally of finite presentation, whence

$$\mathbb{L}_X \in \mathsf{QCoh}(X) = \mathrm{holim}_{\mathrm{Spec}(A)\to X}(A\text{-}\mathsf{mod})$$

is dualisable. The *tangent complex* of X is defined by

$$\mathbb{T}_X \coloneqq \mathbb{L}_X^\vee.$$

It is not enough for X to be Artin.[1] One says X is *locally of finite presentation* if there exists an affine étale cover[2] of X with finitely presented affines. One says $A \in \mathsf{cdga}_{\leq 0}$ is finitely presented if $\mathrm{Hom}_{\mathsf{cdga}_{\leq 0}}(A,-)$ commutes with homotopy colimits, cf. [Stacks, Tag 00QO] for the non-derived case.

[1] Sometimes Artin already means locally of finite presentation as well, but here n-Artin means the quotient of a smooth groupoid of $(n-1)$-Artin stacks, starting with derived schemes in $n=0$. Artin, a.k.a. geometric, means n-Artin for some n. A *smooth (Segal) groupoid* in dSt [Toë14, §3.3] is a (Segal) groupoid object $X_\bullet$ in dSt such that the face maps $X_1 \to X_0$ are smooth morphisms. Its *quotient* is its geometric realisation $|X_\bullet|$.

[2] Calaque requires the following only of a smooth cover. For the dualisability of $\mathbb{L}_X$ this is fine.

Definition 1.5. Let X be Artin and locally of finite presentation.

- Any 2-form $\omega \in \mathcal{A}^2(X, n)$ gives map
$$\omega^\# \colon \mathbb{T}_X \to \mathbb{L}_X[n].$$
It is called *non-degenerate* if $\omega^\#$ is a weak equivalence.
- A *symplectic form of degree* n on X is a point $\omega \in \mathcal{A}^{2,\mathrm{cl}}(X, n)$ whose 2-form component is non-degenerate.

Note that in the non-derived case, weak equivalence reduces to an isomorphism.

2 Unoriented TFTs

Definition 2.1. Let Bord_n denote the category whose objects are closed $(n-1)$-manifolds and whose morphisms are n-dimensional bordisms modulo diffeomorphisms. Disjoint union $\sqcup$ makes it symmetric monoidal.

Definition 2.2. Let Corr denote the category with objects stacks and morphisms
$$\mathrm{Hom}_{\mathsf{Corr}}(X, Y) \coloneqq \{V \to X \times Y\}/\text{w.e.},$$
correspondences modulo weak equivalence. Composition of $V \to X \times Y$ (the constituting coordinate maps) with $W \to Y \times Z$ is given by
$$V \times_Y^{\mathbb{R}} W \to X \times Z,$$
defined up to weak equivalence. Product $\times$ of stacks makes Corr symmetric monoidal.

Proposition 2.3. *Let X be a stack. The assignment*
$$\mathcal{Z}_X \coloneqq \mathbf{Map}((-)_B, X) \colon \mathsf{Bord}_n \to \mathsf{Corr}$$
is (or rather lifts to) an n-TFT, i.e. a symmetric monoidal functor.

Let M be an n-manifold, $\partial M = N \sqcup N'$ with N_1, N_2 closed $(n-1)$-manifolds. Then $\mathcal{Z}_X$ must map M to (the weak equivalence class of) a correspondence $V \to \mathbf{Map}(N_B, X) \times \mathbf{Map}(N'_B, X)$. We may set $V = \mathbf{Map}(M_B, X)$ and use the pullback maps as the two component maps, coming from the Betti versions of the inclusions $N, N' \hookrightarrow M$. Luckily, Betti takes chains instead of cochains, so it is covariant.

Remark 2.4. We must declare $\mathcal{Z}_X(\emptyset) = *$, with $\emptyset$ seen as a closed $(n-1)$-manifold.

Remark 2.5. Note that this is not sensitive to orientation. One might object that Bord_n is not even a category, as M doesn't have a direction, but only hom-sets are defined. Indeed, Corr has the same feature: weak equivalence classes of correspondences are insensitive to the order of X and Y in the product. If you feel disconcerted, you can artificially label morphisms in both Bord_n and Corr with a direction (effectively doubling them), like taking an oriented double cover.

The goal of the rest of this note is to build up to an oriented version, where the input X will be symplectic and the target category will be Lagrangian correspondences.

3 Lagrangian correspondences

Let X and L be Artin and locally of finite presentation, and let ω be an n-shifted symplectic form on X. Recall the following from earlier in the seminar:

Definition 3.1. A *Lagrangian structure* on a map $f \colon L \to X$ is a path h from 0 to $f^*\omega$ (an *isotropic structure*), such that the induced map $\mathbb{T}_f \to \mathbb{L}_L[n-1]$ is a weak equivalence (*non-degeneracy*).

Before promoting Corr to Lagrangian correspondences $\mathsf{LagCorr}$, recall the following silly fact. If (M, ω_M) and (N, ω_N) are symplectic manifolds, then $M \times N$ is again symplectic with symplectic form $\pi_M^* \omega_M + \pi_N^* \omega_N$.

Let now $f_1 \colon L_1 \to X \times \overline{Y}$, $f_2 \colon L_2 \to Y \times \overline{Z}$ be Lagrangian maps, where (X, ω_X) (Y, ω_Y), (Z, ω_Z) are n-shifted symplectic, and $\overline{Y} = (Y, -\omega_Y)$ (same for Z). Again, $X \times Y$ is n-shifted symplectic with form $\pi_X^* \omega_X + \pi_Y^* \omega_Y$ (same for $X \times \overline{Y}$, $Y \times \overline{Z}$).

Proposition 3.2. *The composition $L_1 \times_Y^{\mathbb{R}} L_2 \to X \times \overline{Z}$ has a Lagrangian structure.*

A homotopy $a + b \sim 0$ is a homotopy $a \sim -b$. By assumption, therefore, we have homotopies

$$\begin{aligned} f_1^* \pi_X^* \omega_X &\sim f_1^* \pi_Y^* \omega_Y, \\ f_2^* \pi_Y^* \omega_Y &\sim f_2^* \pi_Z^* \omega_Z, \end{aligned}$$

as well as (cf. the proof of the Lagrangian intersection theorem)

$$\pi_{L_1}^* f_1^* \pi_Y^* \omega_Y \sim \pi_{L_2}^* f_2^* \pi_Y^* \omega_Y \quad \text{in } \mathcal{A}^{2,\mathrm{cl}}(L_1 \times_Y^{\mathbb{R}} L_2, n).$$

(This is the intermediate path in the zero loop that appears on page 10 of Nicola's notes.) In toto, we have a homotopy

$$\pi_{L_1}^* f_1^* \pi_X^* \omega_X \sim \pi_{L_1}^* f_1^* \pi_Y^* \omega_Y \sim \pi_{L_2}^* f_2^* \pi_Y^* \omega_Y \sim \pi_{L_2}^* f_2^* \pi_Z^* \omega_Z.$$

This gives an isotropic structure on the composed correspondence (call it f), which is non-degenerate for silly reasons. Just like in the Lagrangian intersection theorem, start the two defining exact sequences with $\mathbb{T}_f$ and with $\mathbb{L}_{L_1 \times_Y^{\mathbb{R}} L_2}[n-1]$, and give the induced maps between the sequences. Using the assumptions (and that $\mathbb{T} \colon \times \mapsto \oplus$), one sees that the vertical map $\mathbb{T}_f \to \mathbb{L}_{L_1 \times_Y^{\mathbb{R}} L_2}[n-1]$ is sandwiched in a homotopy-commuting diagram with rows exact and all other verticals weak equivalences.

Remark 3.3. In TFT terms, this reduces to an excision statement when $X = Z = *$ (with its zero n-symplectic structure), saying $L_1 \times_Y^{\mathbb{R}} L_2$ is $(n-1)$ symplectic. So, partition functions will be symplectic intersections, whereas time evolution will be given by Lagrangian graphs.

Definition 3.4. Let $\mathsf{LagCorr}_n$ denote the category whose objects are n-shifted stacks and whose morphisms are Lagrangian correspondences up to weak equivalence.

In order to promote $\mathcal{Z}_X$ to an oriented TFT

$$\mathsf{Bord}_n^{\mathrm{or}} \to \mathsf{LagCorr},$$

we must equip the mapping stacks $\mathbf{Map}(N_B, X)$ with symplectic structures and the pullback graphs

$$\mathbf{Map}(M_B, X) \to \mathbf{Map}(N_B, X) \times \overline{\mathbf{Map}(N_B', X)}$$

with Lagrangian structures. Their composability will then follow from the Proposition above.

Indeed, $\mathbf{Map}(Y, X)$, for X n-symplectic, will be symplectic (with a different shift in general) if one can *integrate* on Y, for which one might talk about Y being 'compact' and 'oriented'. Betti stacks of compact oriented manifolds (with or without boundary) will have these properties.

4 Orientation, compactness, transgression

The idea. We are kindly reminded of the following by [Pan+13, §2.1]. Let M be a compact oriented manifold and N a symplectic manifold. Recall that smooth functions $M \to N$ make up a Fréchet manifold $F \coloneqq \mathrm{Map}(M, N)$. Consider

$$M \times \mathrm{Map}(M, N).$$

Forms on M and N can be pulled back to $M \times F$ along the projection $\pi_M : M \times F \to M$ and the evaluation $\mathrm{ev} : M \times F \to N$ (which is smooth essentially by definition). Further, given a form $\eta \in \Omega^p(M \times F)$, we may integrate out M to get a form on F:

$$\int_M \eta \in \Omega^{p-\dim(M)}(F).$$

In particular, for N symplectic with form ω, one can consider

$$\eta \coloneqq \pi_M^*(1) \wedge \mathrm{ev}^*(\omega) \in \Omega^{0+2}(M \times F),$$

and evaluate

$$\widetilde{\omega} \coloneqq \int_M \eta \in \Omega^{2-\dim(M)}(F).$$

This is said to make F symplectic. Even if you are not satisfied with this being a symplectic form, this idea works perfectly in the stacky context.

Remark 4.1 (Coda – read at the end). As Artem mentioned, you would normally pullback the volume form on M and not the function 1, and would then get a 2-form on F. The way PTVV present it is due to the way they define orientation, which integrates functions, not top-forms. But this is misleading: even in the Betti case these functions are $\mathrm{Sing}^\bullet(M, \Bbbk)$, and the orientation class pairs with $\mathrm{Sing}^{\dim(M)}(M, \Bbbk$, giving a $\dim(M)$-orientation. So the fully analogous thing to do in the differential geometry context would indeed be to pullback the volume form, not 1. The shift by $-\dim(M)$ in 4.6 is thus not quite the $-\dim(M)$ in $\widetilde{\omega}$ above: the form degree doesn't shift, but the integral degree shifts due to top cochains in $\mathbb{R}\Gamma(M_B)$ being in degree $\dim(M)$.

Notation 4.2. We write

$$\mathbb{R}\Gamma(X, -) : \mathsf{QCoh}(X) \to \Bbbk\text{-}\mathsf{mod},$$

or $\mathbb{R}\Gamma(-)$ when X is understood, for the forgetful functor induced by the forgetful functors $A\text{-}\mathsf{mod} \to \Bbbk\text{-}\mathsf{mod}$ on open affines. It is the derived-∞ version of global sections.

Definition 4.3. A stack Σ over $\mathrm{Spec}(A)$ is called *$\mathcal{O}$-compact over* $\mathrm{Spec}(A)$ if

- $\mathcal{O}_\Sigma \in \mathsf{QCoh}(\Sigma)$ is a *compact object*, i.e. $\mathsf{QCoh}(\mathcal{O}_\Sigma, -)$ preserves (countable) homotopy-colimits (cf. nLab), and
- for any $E \in \mathsf{Perf}(\Sigma)$, $\mathbb{R}\Gamma(E)$ is a perfect A-module.[3]

We say a stack Σ over $\Bbbk$ is *$\mathcal{O}$-compact* if $\Sigma_A \coloneqq \Sigma \times \mathrm{Spec}(A)$ is $\mathcal{O}$-compact over $\mathrm{Spec}(A)$ for any A.

The point is, for any other stack Y, there is a natural map

$$\mathrm{DR}^\bullet(\Sigma \times Y) \to \mathbb{R}\Gamma(\mathcal{O}_\Sigma) \otimes \mathrm{DR}^\bullet(Y),$$

locally induced by PTVV's affine Künneth formula and then by projecting to the function part. Künneth glues mod w.e., and the projection glues mod w.e. by the compactness of $\mathcal{O}_\Sigma$. We have a restricted (induced) a map

$$\mathcal{A}^{p(,\mathrm{cl})}(\Sigma \times Y) \to \mathbb{R}\Gamma(\mathcal{O}_\Sigma) \otimes \mathcal{A}^{p(,\mathrm{cl})}(Y).$$

Think of these decompositions as a Fubini theorem (for the moment pre-integration), so compactness corresponds morally to the 'σ-finiteness' of $\Sigma \times Y$ and the integrals over $\Sigma \times Y$ being finite.

It remains to integrate out the Σ-factor.

[3][Pan+13] specifically says perfect A-module, while in [Cal14]; [Cal15] it seems to be perfect $\Bbbk$-module.

Definition 4.4. An *m-orientation* on Σ is a map

$$[\Sigma]\colon \mathbb{R}\Gamma(\mathcal{O}_\Sigma) \to \Bbbk[-m]$$

such that, for any $E \in \mathsf{Perf}(\Sigma)$, the pairing

$$\mathbb{R}\Gamma(E) \otimes \mathbb{R}\Gamma(E^\vee) \to \mathbb{R}\Gamma(\mathcal{O}_\Sigma) \to \Bbbk[-m]$$

is *non-degenerate*. This means that the adjoint map

$$\mathbb{R}\Gamma(\Sigma_A, E) \to \mathbb{R}\Gamma(\Sigma_A, E^\vee)^\vee[-d]$$

is a weak equivalence for any $E \in \mathsf{Perf}(\Sigma_A)$ and any $A \in \mathsf{cdga}_{\leq 0}$.

The notation $[\Sigma]$ suggests that we think of this as integration against the fundamental class (the choice of which is an orientation, after all). The pairing on gauge fields can be thought of as a generalised L^2-pairing, which first pairs in the fibres and then integrates against $[\Sigma]$. The non-degeneracy is a weak Hilbert condition.

Notation 4.5. For Σ m-oriented and $\mathcal{O}$-compact, let

$$\int_{[\Sigma]} \colon \mathcal{A}^{p(,\mathrm{cl})}(\Sigma \times Y) \to \mathcal{A}^{p(,\mathrm{cl})}(Y)[-d]$$

denote the composition of the Fubini map and $[\Sigma] \otimes \mathrm{id}$.

In particular for Σ as above, for $Y = \mathbf{Map}(\Sigma \times Y)$ and for some $\omega \in \mathcal{A}^{p,\mathrm{cl}}(X, n)$, we have

$$\int_{[\Sigma]} \mathrm{ev}^*\omega \in \mathcal{A}^{p,\mathrm{cl}}\left(\mathbf{Map}(\Sigma, X), n-m\right).$$

Proposition 4.6 (Transgression)**.** *Let Σ, X and ω be as above. If ω is symplectic, $\mathbf{Map}(\Sigma, X)$ is $(n-m)$-symplectic.*

The non-degeneracy of $\int_{[\Sigma]} \mathrm{ev}^*\omega$ follows straightforwardly from the non-degeneracy of the orientation, in the special case of a symplectic pairing.

5 Oriented TFTs

It remains to transgress in the case with boundary.

Let Σ, Σ' be $\mathcal{O}$-compact, and $\phi\colon \Sigma \to \Sigma'$ be a map. Let $[\Sigma]$ be an m-orientation on Σ, so that we an induced map

$$\phi_*[\Sigma]\colon \mathbb{R}\Gamma(\Sigma', \mathcal{O}_{\Sigma'}) \to \Bbbk[-m].$$

Definition 5.1. A *boundary structure* on ϕ is a path from $\phi_*[\Sigma]$ to 0.

Of course, there is a natural non-degeneracy condition on boundary structures, which is built so that our TFTs will land in *Lagrangian* correspondences and not just isotropic ones. See [Cal15, §2.2.3].

Let X be a stack and $\phi\colon \Sigma \to \Sigma'$ be as above. Write

$$\mathrm{ev}'\colon \Sigma' \times \mathbf{Map}(\Sigma', X) \to X$$

and

$$\int_{\phi_*[\Sigma]} \colon \mathcal{A}^{p,\mathrm{cl}}(\Sigma' \times \mathbf{Map}(\Sigma', X), n) \to \mathcal{A}^{p,\mathrm{cl}}(\mathbf{Map}(\Sigma', X), n-d).$$

Non-degenerate boundary structures are mapped to Lagrangian structures. In symbols:

Proposition 5.2. *if $\omega \in \mathcal{A}^{2,\mathrm{cl}}(X,n)$ is a symplectic form on X, the pullback*

$$\phi^* \colon \mathbf{Map}(\Sigma', X) \to \mathbf{Map}(\Sigma, X)$$

has a Lagrangian structure, with the path from $\int_{\phi_*[\Sigma]} \mathrm{ev}'\omega$ to 0 induced by the boundary structure.

The non-degeneracy of the isotropic structure is massaged into the non-degeneracy of the boundary condition; see the proof of [Cal15, Theorem 2.9]. Let now X be m-shifted symplectic

Proposition 5.3. *The assignment*

$$\mathcal{Z}_X \coloneqq \mathbf{Map}((-)_B, X) \colon \mathsf{Bord}_n^{\mathrm{or}} \to \mathsf{LagCorr}_{m-(n-1)}$$

is a symmetric monoidal functor.

Betti stacks coming from compact oriented manifolds (possibly with boundary) come with the expected data for compactness, orientation and (non-degenerate) boundary structures; see [Cal15, §3.1]. Let us only mention the following to motivate the boundary data: If N is oriented and closed, its fundamental class $[N]$ gives an evaluation map $\mathrm{Sing}^\bullet(N, \Bbbk) \to \Bbbk[-\dim(N)]$. Note that $\mathbb{R}\Gamma(\mathcal{O}_{N_B}) = \mathrm{Sing}^\bullet(N, \Bbbk)$. Lastly, when M is oriented with boundary ∂M, its fundamental class $[M]$ is in *relative* top-degree homology, and is sent to $[\partial M]$ by the boundary map, which leads to a non-degenerate boundary structure under Betti. Al coda 4.1.

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