# What is... a Drinfeld associator?

Ödül Tetik

Zurich Graduate Colloquium
December 1st, 2020

# Table of contents

# Setting: what is an associator?

## What is an associator?

Say you are in category $\mathcal{C}$, and there is a product map

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

Think:

- rings, algebras...
- $\mathcal{V}\mathrm{ect}$, or the module category of a ring
- the representation category of a group/field/Lie algebra
- (virtual) bundles over a manifold
- modules over a sheaf on a scheme
- state spaces of point-particles in a discrete quantum-mechanical system
- ...

Even in $\mathcal{V}\mathrm{ect}$, the usual tensor product is neither commutative nor associative.

Instead of commutativity, commutativity isomorphisms:

$$R_{A,B} \colon A \otimes B \xrightarrow{\sim} B \otimes A.$$

Similarly, associativity isomorphisms:

$$\Phi_{ABC} \colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C).$$

Immediate problem: if we take different paths between the same domain and target, will those paths be the same?

In symbols, say

$$(A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \rightrightarrows (A_3 \otimes A_1) \otimes (A_4 \otimes A_2)$$

are two maps using only $R$'s and $\Phi$'s. Are they the same?

If so, $(\mathcal{C}, \otimes, R, \Phi)$ is called coherent.

A priori, one has to check an infinite number of diagrams.

Mac Lane's coherence theorem reduces this to checking only 3 diagrams:

Two hexagons:

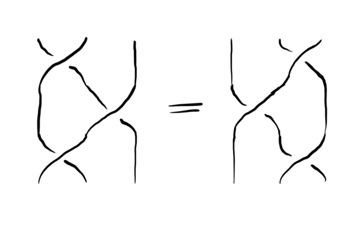
$$
\begin{tikzcd}
 & A(BC) \arrow[r] & (BC)A \arrow[dr] & \\
(AB)C \arrow[ur] \arrow[dr] \arrow[rrr, dotted] & & & B(CA) \\
 & (BA)C \arrow[r] & B(AC) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (AB)C \arrow[r] & C(AB) \arrow[dr] & \\
A(BC) \arrow[ur] \arrow[dr] \arrow[rrr, dotted] & & & (CA)B \\
 & A(CB) \arrow[r] & (AC)B \arrow[ur] &
\end{tikzcd}
$$

and a pentagon:

$$
\begin{tikzcd}
 & A(B(CD)) \arrow[dl] \arrow[dr] \arrow[ddr, dotted] & \\
A((BC)D) \arrow[d] & & (AB)(CD) \arrow[d] \\
(A(BC))D \arrow[rr] & & ((AB)C)D
\end{tikzcd}
$$

## Terminology, instances

- We are skipping unit axioms, but these are usually fine.
- If $R$ is an involution, checking one hexagon suffices. Then: symmetric monoidal. Otherwise: braided monoidal, with braiding $R$.
- '$R$': scattering matrices in quantum statistical mechanics, a.k.a. '$R$-matrices'. The hexagon $=$ Yang–Baxter equation.
- 'Braiding': braid groups: $R_{A,B}$ crosses strand $A$ over $B$. The (positive) hexagon:



- No $\Phi$ with braids if they are not parenthesised. (They just sit next to each other, ungrouped.) We will come back to this!

# Perspective 1: quantum groups

## Prelude: Hopf algebras

Algebraisation of geometry: replace space $X$ by a set $C(X)$ of functions.

Automatic algebra structure on $C(X)$ if it is nice:

- Functions can be added and multiplied pointwise,
- so $C(X)$ is a $k$-algebra for $k$ the target field of the functions (interpret scalars as constant functions).

Each map $X \to Y$ has its pullback $C(Y) \to C(X)$ in the opposite direction. E.g. an inclusion map induces a restriction map on functions / sheaves are contravariant.

Thus, structure on $X$ translates to opposite structure on $C(X)$.

If $X = G$ is a group, the multiplication map

$$\mu \colon G \times G \to G$$

induces a comultiplication map $\Delta = \mu^* \colon C(G) \to C(G \times G)$. In nature, there will be a $\otimes$ so that comultiplication looks like

$$\Delta \colon C(G) \to C(G) \otimes C(G).$$

If $\mu$ and $\Delta$ on an associative algebra $A$ interact as in the case $A = C(G)$, then $A$ is called a bialgebra. When in addition there is an involution $A \to A$ acting like the (pullback of the) inversion in $G$, then $A$ is called a Hopf algebra.

One also has an augmentation $\varepsilon\colon A \to k$ landing in the ground field $k$, morally coming from evaluation at the identity element of the group.

The representation category of a bialgebra $A$ is monoidal via $\Delta$:

$$a \cdot (v_1 \otimes v_2) \coloneqq (\Delta(a)_1 \cdot v_1) \otimes (\Delta(a)_2 \cdot v_2) \in V_1 \otimes V_2$$

where $\Delta(a)_{1/2} \in A$ denotes the first/second factor of $\Delta(a) \in A \otimes A$.

## Relaxing commutativity

Let $\underline{1}, \underline{2}$ be $A$-modules. There are 2 ways to act with $A$ on $\underline{1} \otimes \underline{2}$, namely

$$A \xrightarrow{\Delta} A \otimes A$$

versus

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\sigma} A \otimes A,$$

denoted by $\Delta'$, where $\sigma$ swaps the factors.

Assume we have an internal commutativity $R$, i.e. (by abuse of notation) $R \in A \otimes A$, so that

$$\Delta' = R \Delta R^{-1}.$$

This $R$ then (globally) realises the isomorphism(s)

$$\underline{1} \otimes \underline{2} \simeq \underline{2} \otimes \underline{1}.$$

# Relaxing associativity

Let $\underline{3}$ be another $A$-module.

Similarly, there are 2 ways to act with $A$ on '$\underline{1} \otimes \underline{2} \otimes \underline{3}$' (without using $\sigma$ or $R$), namely

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\Delta \otimes \mathrm{id}} A \otimes A \otimes A$$

versus

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id} \otimes \Delta} A \otimes A \otimes A.$$

The corresponding $A$-modules are written $(\underline{1} \otimes \underline{2}) \otimes \underline{3}$ and $\underline{1} \otimes (\underline{2} \otimes \underline{3})$.

We may similarly ask for an internal $\Phi \in A \otimes A \otimes A$ such that

$$(\mathrm{id} \otimes \Delta)\Delta = \Phi((\Delta \otimes \mathrm{id})\Delta)\Phi^{-1},$$

globally realising

$$(\underline{1} \otimes \underline{2}) \otimes \underline{3} \simeq \underline{1} \otimes (\underline{2} \otimes \underline{3}).$$

# Relaxing commutativity and associativity

In Hopf algebras: $\Phi = 1_{A\otimes A\otimes A}$, $R = 1_{A\otimes A}$.

If we allow a $\Phi$, we have a quasi-Hopf algebra. If we also allow an $R$, we have a quasitriangular quasi-Hopf algebra.

We won't be much interested in an antipode $A \to A$, so we can just talk about bialgebras and their weak versions.

For coherence, we must impose the hexagons and the pentagon. E.g., the pentagon translates to

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\cdot(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi) = (1_A\otimes\Phi)\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\cdot(\Phi\otimes 1_A),$$

in $A^{\otimes 4}$. The hexagons will feature both $R$ and $\Phi$.

There is some more minor structure that we are skipping for the moment ($\varepsilon$, behaviour on scalars, etc.).

# Quantization

A central phenomenon in QP is that action is quantized. We have a unit of action, called Planck's constant, denoted by $\hbar$.

For classical systems (meso-/large-scale) ($\sim$ low energies), effectively $\hbar = 0$, whereas in small scales ($\sim$ high energies) $\hbar \neq 0$.

Another central phenomenon in QP: while in CP the algebra of observables is a commutative algebra, with dynamics encoded by a Poisson structure $\{-, -\}$, in QP, observables make up only an associative algebra. Dynamics is encoded by the ordinary commutator $[-, -]$ instead.

For $A_0$ a commutative algebra over $k$, an associative algebra $A$ over $k[[\hbar]]$ such that

$$A/\hbar A = A_0$$

is called a quantization of $A_0$.

Given $A$, we can define a Poisson bracket on $A_0$ by

$$\{a \pmod{\hbar}, b \pmod{\hbar}\} := \frac{1}{\hbar}[a, b] \pmod{\hbar},$$

which is the $\hbar^1$-term of $[a, b]$.

Conversely, given a *Poisson* algebra $A_0$, a quantization is an associative algebra $A$ over $k[[\hbar]]$ with $A/\hbar A = A_0$ such that $[-, -]$ on $A$ and $\{-, -\}$ on $A_0$ are related by the above formula.

# Central example

Let $\mathfrak{g}_0$ be a Lie algebra over a field $k$, and

$$U\mathfrak{g}_0 = \left(\bigoplus_{l \geq 0} \mathfrak{g}_0^{\otimes l}\right) / \left(a \otimes b - b \otimes a \sim [a, b]\right)$$

its universal enveloping algebra.

Comultiplication:

$$\Delta(1) = 1 \otimes 1$$

on the $k$-component ($\mathfrak{g}_0^{\otimes 0} = k$), and on the rest by Leibniz:

$$\Delta(a) = a \otimes 1 + 1 \otimes a.$$

This is cocommutative and coassociative.

We now ask for a Lie algebra $\mathfrak{g}$ over $k[[\hbar]]$, isomorphic as a topological $k[[\hbar]]$-module to some $V[[\hbar]]$ with $V$ a $k$-vector space. We ask that

$$\mathfrak{g}/\hbar\mathfrak{g} = \mathfrak{g}_0,$$

and consider

$$A = \hat{U}\mathfrak{g},$$

the $\hbar$-adic completion of $U\mathfrak{g}$.

We now look for $R$ and $\Phi$ on $A$.

Assume $R$ symmetric and $\mathfrak{g}$-invariant. Then there will exist some symmetric $\mathfrak{g}$-invariant $t \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$R = e^{\hbar t/2},$$

so we may construct $R$ from some such $t$ without restriction.

# Drinfeld's theorem

Drinfeld, building on earlier foundational work of Kohno, showed in the late 80's that given any such $R$, there exists a $\Phi$ (called a Drinfeld associator) satisfying all the coherence conditions.

Moreover, $\Phi$ is unique up to gauge-invariance, which can be defined purely algebraically.

To construct a $\Phi$, he translated the commutation of the coherence diagrams to the flatness of the Knizhnik–Zamolodchikov (KZ) connection, a basic object in conformal field theory. This particular $\Phi_{KZ}$ is known as the KZ associator. (More on this later.) Later, other associators have also been constructed using geometric methods.

This yielded a 'universal formula' for $\Phi_{KZ}$ with $\mathbb{C}$ coefficients.

In a subsequent paper he showed that the Grothendieck–Teichmüller group acts freely and transitively on the set of associators.

A rational associator was also obtained.

The absolute Galois group of $\mathbb{Q}$ (as well as its motivic version) can be embedded into the Grothendieck–Teichmüller group. Thus, much of number theory is hidden in the set of Drinfeld associators.

## Overview of two other main perspectives

**Knot theory**: Bar-Natan gave an equivalent definition of a Drinfeld associator as an equivalence between the categories of braids and chord diagrams (that preserves some extra natural structure). (The connection to knot theory was already explicit in earlier work of Kohno.) Thus, any Drinfeld associator gives a universal Vassiliev invariant. The Kontsevich integral, a universal Vassiliev invariant, is a direct recasting of the KZ associator. Even better, this is tightly connected to determining (homotopy types of) more general embedding spaces and Goodwillie–Weiss calculus.

**Quantization**: One may similarly ask for quantizations of a Poisson algebra given by the algebra of functions on a Poisson manifold $M$: the (global) algebra of observables of a classical scalar field theory. Kontsevich and later Tamarkin, the latter building on earlier work of Etingof–Kazhdan, and in two different approaches, showed that any Drinfeld associator gives such a quantization in the local case ($M = \mathbb{R}^n$). Kontsevich also gave an independent construction. Even this case was open since the early days of QFT. Generalisations abound, but are missing in some very important cases.

## What the f?

In a recent paper, Furusho has shown that in important cases ($\sim$ in Drinfeld's original setting) the pentagon already implies the hexagons.

(Also, Bar-Natan purports to have given a simpler proof of Furusho's theorem.)

I have not yet studied this and don't know how to interpret it.

# Perspective 2: knot (Vassiliev) theory

# Definitions

The braid group $B_n$ on $n$ strands is generated by the crossings: write simply $\sigma_i$ $(1 \leq i < n)$ for the crossing of strand $i$ over $i+1$. By a result of Alexander, any knot is the closure of a braid.

There are two relations:

- Causality/spatial independence:
$$\sigma_i \sigma_j = \sigma_j \sigma_i$$
  for $|i - j| > 1$;
- The hexagon:
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

We may view a braid as a morphism in a category whose objects are finite collections of points.

Better, we may make objects parenthesised collections of points, or equivalently, introduce a notion of distance between the points. We may then consider parenthesised braids as morphisms.

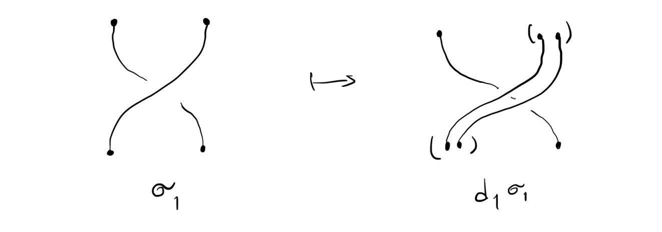
We no longer fix the number of strands.

# Face and degeneracy maps

We may also consider formal linear combinations (with coefficients from some algebra) of braids as morphisms. This category, PaB, has more structure and relations.

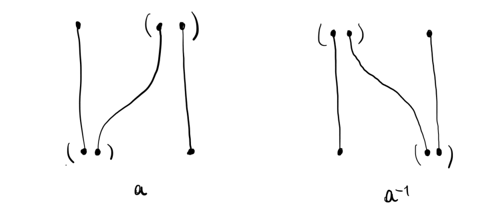
First, the face and degeneracy maps, denoted by $s_i$ and $d_i$, respectively, that act on braids:

- $s_i$ deletes strand $i$;
- $d_i$, on a braid $b$ with $n$ strands, for $1 \leq i \leq n$, doubles strand $i$:

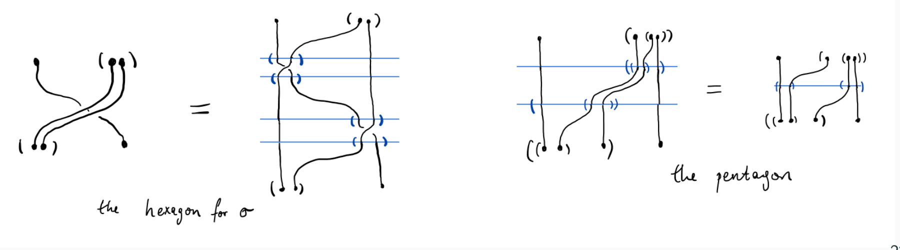



Also, $d_0$ adds a straight strand on the left, $d_{n+1}$ adds one on the right.

PaB also includes the associators $a^{\pm 1}$:

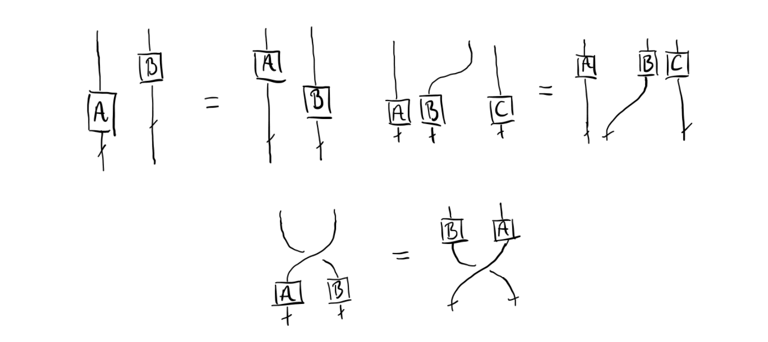



So we want the parenthesised versions of the usual relations:



# Locality relations

Further, a full description of PaB would include the following locality relations:



# Chord diagrams

Paranthesised braids have a kind of linearisation, called chord diagrams. A structure-preserving equivalence between braids and chords is equivalent to a Drinfeld associator.

One way that we can talk about a 'linearisation' is vague but geometric: the braid algebra (restricted to $n$ strands) is isomorphic to

$$\pi_1(\mathrm{Conf}_n(\mathbb{C})),$$

while the algebra of chord diagrams (restricted to $n$ strands) is essentially the cohomology algebra

$$H^*(\mathrm{Conf}_n(\mathbb{C})).$$

V. Arnold gave a generators-relations presentation of this cohomology in 1969.

The result is that the cohomology is generated by 1-forms

$$\omega_{ij} = \mathrm{d}\log(z_i - z_j) = \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j} \quad (1 \le i, j \le n)$$

that satisfy, three relations (with $\wedge$ as product):

- symmetry: $\omega_{ij} = \omega_{ji}$;
- locality: $[\omega_{ij}, \omega_{kl}] = 0$ for $i, j, k, l$ all different;
- 4T or Arnold's relations: $[\omega_{jk}, \omega_{ij} + \omega_{ik}] = 0$ for $i, j, k$ all different.

These $\omega_{ij}$ will appear again in the KZ connection/associator.

The name '4T' is no coincidence: our chord diagram algebra will indeed be the chord diagram algebra for finite-type Vassiliev invariants. It is isomorphic to the associated graded of $\mathrm{PaB}$ with respect to the 'augmentation ideal' $I$ containing the (combinations of braids) $\sum a_i B_i$ such that $\sum a_i = 0$ in the base algebra.

The 4T relation can be interpreted as a linear version of the hexagon. In a way, the hexagon is the 'Lie group' version, and 4T is the induced relation in the 'Lie algebra'. (More on this below.)

Let us now define $\mathrm{PaCD}$. Its objects are the same as $\mathrm{PaB}$, i.e. parenthesisations. Its morphisms are formal linear combinations of expressions

$$D \cdot P$$

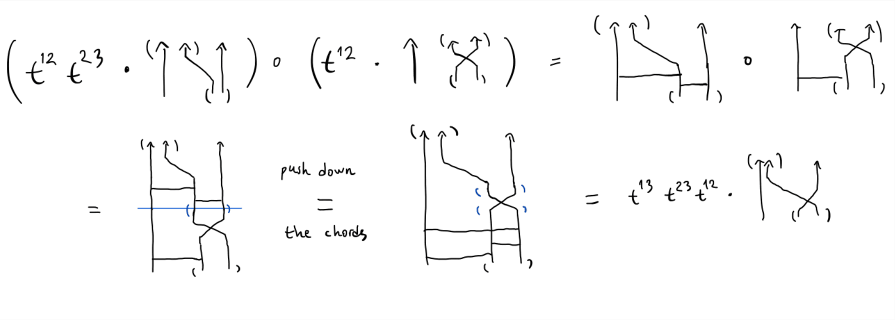
where

- $P$: parenthesised permutation (source and target fixed, say with $n$ points);
- $D$: from the algebra $\mathcal{A}_n^{\mathrm{pb}}$ over some base algebra (most importantly $\mathbb{C}$ or $\mathbb{Q}$), generated by $t^{ij}$, $1 \leq i \neq j \leq n$, satisfying the three relations above.

Composition is a bit unnatural at first sight, because the chords $t_{ij}$ are not ordered in time inside the permutation $P$, in contrast to the crossings in $\mathrm{PaB}$.

Let us mention the example Bar-Natan gives.

The convention is to push down the chords after composing on the nose. This changes the $i, j$ in $t^{ij}$.




Caution: I follow the opposite ordering convention to Bar-Natan's.

# Final structure

There are related face and degeneracy $s_i, d_i$ maps on $\mathrm{PaCD}$.

There is a natural coproduct on braids

$$\square\colon \mathrm{PaB} \to \mathrm{PaB} \otimes \mathrm{PaB}$$

with $\otimes$ naturally defined. Also, a coproduct

$$\square\colon \mathrm{PaCD} \to \mathrm{PaCD} \otimes \mathrm{PaCD}$$

is defined by

$$\square t^{ij} = t^{ij} \otimes 1 + 1 \otimes t^{ij}.$$

Lastly, one considers natural completions

$$\widehat{\mathrm{PaB}} \quad \text{and} \quad \widehat{\mathrm{PaCD}}.$$

Let $\hat{B}$ and $\hat{C}$ denote $\widehat{\mathrm{PaB}}$ and $\widehat{\mathrm{PaCD}}$ with all the extra structure, including the braidings (next slide for the braiding in chords).

# Braiding in chords

Clearly, $\mathrm{PaCD}$ is generated (via the $d_i$), in suggestive notation, by

$$a^{\pm}, X, H.$$

We specify the braiding in chords as

$$\tilde{R} = \exp(-\frac{1}{2}H) \cdot X$$

so that $\tilde{R}^2 = \exp(H)$.

We will require a structure-preserving functor

$$\hat{Z} \colon \hat{B} \to \hat{C}$$

to send

$$\sigma \mapsto \tilde{R}.$$

*(There are good reasons for this, but I cannot go into it.)*

# Equivalences

There are forgetful functors $\hat{B}, \hat{C} \to \mathrm{PaP}$ that take the underlying parenthesised permutations.

An equivalence is a *structure-preserving* functor

$$\hat{Z} \colon \hat{B} \to \hat{C}$$

*over* $\mathrm{PaP}$.

It is 'easy' to see that such a $\hat{Z}$ is determined by its value on $a$:

$$\hat{Z}(a) = \Psi_Z \cdot a$$

where $\Psi_Z \in \mathcal{A}_3^{pb}$.

One can see that the equations this $\Psi_Z$ has to satisfy mean that this is equivalent to giving a Drinfeld associator.

# Perspective 3: Kontsevich formality/Deligne's conjecture

Because talking about this will require a lot of background, let me just say the following.

Kontsevich's deformation quantization theorem is a (not exactly trivial) consequence of his formality theorem. We will look at the affine algebraic case.

Let $A$ be a polynomial algebra, $HC^*(A; A)$ its Hochschild cochain complex with values in $A$,

$$HC^n(A; A) = \mathrm{Hom}(A^{\otimes n}, A)$$

with the differential at degree $n$ given by

$$(-1)^n(df)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) - \sum_0^{n-1} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^{n-1} f(a_0, \dots, a_{n-1}) a_n.$$

This is the dual of the more familiar homological version. The cohomology is denoted by $HH^*(A; A)$.

# Gerstenhaber structure

One can identify $HC^*[1]$ with the coderivations of the cofree coalgebra cogenerated by $A[1]$. Indeed, $HC^*[1]$ is a dg Lie algebra. The differential is induced by the multiplication $\mu$ on $A$.

Even better, there is a multiplication $m$ on $HC^*$:

$$m(x \otimes y) = \mu \circ (x \boxtimes y).$$

In *cohomology*, these structures make $HH^*$ a Gerstenhaber algebra. That is, the induced multiplication is commutative and associative, and the induced bracket is a derivation with respect to the induced multiplication.

This is also called a $P_2$-algebra structure on $HH^*$, i.e. a Poisson structure where the bracket has degree $-1$.

# Formality, Deligne's conjecture

The formality theorem states that $HC^*$ is formal, i.e. in the homotopy category of dg Lie algebras it is isomorphic to its cohomology $HH^*$.

Now, Tamarkin proved that the chains operad $C_*E_2$ of little 2-disks is formal, and by F. Cohen's earlier work we know that $H_*E_2 \simeq P_2$, so

$$C_*E_2 \simeq P_2.$$

This can be used to prove Deligne's conjecture, that the Gerstenhaber structure descends from a $C_*E_2$-structure.

Once one has this, the formality theorem can be proved (as was done by Tamarkin) using Halperin–Stasheff theory, an obstruction theory for formality, and some ideas from operadic Koszul duality.

Choosing an underlying homotopy-algebra structure depends on the choice of a Drinfeld associator (as the construction goes through Etingof–Kazhdan quantization theory).

# Perspective 4: coherence in higher categories/combinatorics

A few words on Kapranov, Voevodsky, Manin–Schechtman braids, etc.

# Geometric construction

Let $R = e^{\hbar t/2}$ where $t \in \mathfrak{g} \otimes \mathfrak{g} \subset (U\mathfrak{g})^{\otimes 2}$ is symmetric and $\mathfrak{g}$-invariant. We are looking for an associator $\Psi$.

Morally, $\Psi$ should 'pass from the parenthesisation 1(23) to (12)3'. Drinfeld's idea was to replace different parenthesisations by asymptotic zones, as follows.

Consider the following differential equation in $(U\mathfrak{g})^{\otimes 3}[[\hbar]]$, $x \in (0,1)$:

$$G'(x) = \hbar \left( \frac{t^{12}}{x} + \frac{t^{23}}{x-1} \right) G(x)$$

where $t^{12} := t \otimes 1$, $t^{23} = 1 \otimes t$, and we impose the asymptotic behaviour

$$\begin{aligned} G_1(x) &\sim x^{\hbar t^{12}} && \text{for } x \to 0 \\ G_2(x) &\sim (1-x)^{\hbar t^{23}} && \text{for } x \to 1. \end{aligned}$$

The idea is to define $\Psi$ by requiring

$$G_1 = G_2 \Psi.$$

This is easily shown to determine a constant $\Psi$.

# The KZ connection

One must check that this $\Psi$ satisfies the defining relations, including the pentagon and the two hexagons with respect to our $R$.

For example, the pentagon will concern various images (via $\Delta$) of $\Psi$ in $(U\mathfrak{g})^{\otimes 4}[[\hbar]]$. The generalisation of the system above reads

$$\partial_i W = \hbar \sum_{j \neq i} \frac{t^{ij}}{z_i - z_j} W \qquad i = 1, 2, \ldots, n$$

with $W(z_1, \ldots, z_n) \in (U\mathfrak{g})^{\otimes n}[[\hbar]]$ and real $z_i$.

Originally, this comes from the connection

$$\nabla_{KZ} = d - (coeff) \sum t^{ij} \frac{dz_i - dz_j}{z_i - z_j},$$

called the KZ connection, so the system says that $W$ is a horizontal path with respect to $\nabla_{KZ}$.

In the setting of CFT, $\nabla_{KZ}$ is a connection on the so-called *conformal block bundle* over $\mathrm{Conf}_n(\mathbb{C})$ or $\mathrm{Conf}_n(\mathbb{C}P^1)$.

The point of all this is that $\nabla_{KZ}$ is flat (Drinfeld calls it self-consistent – this is what that means), i.e. its curvature vanishes.

This can be shown directly using the Arnold relations. More specifically, the Arnold relations imply, writing $\nabla_{KZ} = d - \omega$, that

$$\omega \wedge \omega = 0.$$

Combining this with the easy $d\omega = 0$, it follows that $\nabla_{KZ}$ is flat, since its curvature is

$$d\omega + \omega \wedge \omega.$$

THIS IS ABSOLUTELY FANTASTIC NEWS.

# E.g. the pentagon

To check the pentagon, we set $n = 4$. Consider the system over the domain

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 | z_1 > z_2 > z_3 > z_4\},$$

where we distinguish five asymptotic zones that correspond to that five parenthesisations:

$$\begin{aligned}
&z_1 - z_2 \ll z_1 - z_3 \ll z_1 - z_4 && ((12)3)4\\
&z_2 - z_3 \ll z_1 - z_3 \ll z_1 - z_4 && (1(23))4\\
&z_2 - z_3 \ll z_2 - z_4 \ll z_1 - z_4 && 1((23)4)\\
&z_3 - z_4 \ll z_2 - z_4 \ll z_1 - z_4 && 1(2(34))\\
&z_1 - z_2 \ll z_1 - z_4,\ z_3 - z_4 \ll z_1 - z_4 && (12)(34).
\end{aligned}$$

One can generalise the asymptotic behaviours of $G_1, G_2$ to define solutions $W_1, \ldots, W_5$ with clever asymptotic behaviours.

The $W_i$ thus defined, it is a direct algebraic check that we have

$$W_1 = W_2(\Psi \otimes 1) \qquad W_1 = W_5(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Psi)$$
$$W_2 = W_3(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Psi) \qquad W_5 = W_4(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Psi)$$
$$W_3 = W_4(1 \otimes \Psi)$$

Now the flatness of $\nabla_{KZ}$ implies that whichever way one passes say from $W_1$ to $W_5$, the result will be the same.

Indeed, an equivalent definition of flatness is that holonomy depends only on the homotopy class of the chosen path. Thus, *the commutation of the pentagon is translated to the homotopy-invariance of the holonomy of a flat connection.* The hexagons can be checked similarly.

So, $\Psi$ is indeed a Drinfeld associator.

# Further directions

- Higher gauge-theoretic analogues of the KZ construction (contains issues about the connection)
- Higher braids...
- Shifted Poisson structures / analogues of chord diagrams?
- 'Extended quantization': what if the manifold has boundaries, corners... and defects, etc.?(ongoing work)
- Are there analogues of Grothendieck–Teichmüller? Belyi's theorem? $\rightarrow$ do 'higher dimensional' absolute Galois groups embed? Motivic stuff?
- ...

# Annotated references

# References i

I will only give references on the topics that I could mention during the talk.

Some papers of Drinfeld's about these things:


- Drinfeld (1986), 'Quantum groups', ICM Proceedings.
- Drinfeld (1989), 'Quasi-Hopf algebras and Knizhnik–Zamolodchikov equations', in Belavin et al. (eds.), *Problems of Modern Quantum Field Theory*.
- Drinfeld (1990), 'Quasi-Hopf algebras', *Leningrad Math. J.*
- Drinfeld (1991), 'On quastriangular quasi-Hopf algebras and a group closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$', *Leningrad Math. J.*


Kohno's work came very shortly before; see also his book, and Tsuchiya–Kanie:


- Kohno (1987), 'Monodromy representations of braid groups and Yang–Baxter equations', *Ann. Inst. Fourier (Grenoble)*.
- Kohno (1988), 'Quantized universal enveloping algebras and monodromy of braid groups'.
- Tsuchiya & Kanie (1988), 'Vertex operators in conformal field theory on $\mathbb{P}^1$ and monodromy representations of braid groups'.
- Kohno (2002), *Conformal field theory and topology*, AMS.

For the braids-chord diagrams picture, see

- Bar-Natan (1998), 'On associators and the Grothendieck-Teichmuller group, I', *Selecta Math.*

Recently, Fresse wrote a nice related survey, that also touches on number-theoretic developments:

- Fresse (2019), 'Little discs operads, graph complexes and Grothendieck-Teichmüller groups', in Miller (ed.), *Handbook of Homotopy Theory*.

Furusho's paper on the pentagon implying the hexagons in certain settings:

- Furusho (2010), 'Pentagon and hexagon equations', *Ann. Math.*

A useful exposition (and in part expansion) of Tamarkin's proof of Kontsevich formality was written up by Hinich. He also explains the basic ideas needed from Koszul duality, homology perturbation theory, Halperin–Stasheff obstruction theory, etc.

- Hinich (2003), 'Tamarkin's proof of Kontsevich formality theorem', *Forum Math.*