

Geometric factorisation algebras and Morita theory

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24 June 2022

Defects and Symmetry
King's College London

Duality: factorisation homology \leftrightarrow functorial field theories?

FH: homology theories at chain level:

- ▶ **input:** disk-algebra A fitting the geometry of spacetime X ,
e.g. a framed n -disk-algebra for framed n -manifold X
Nonphysical: $A \equiv \mathbb{Z}$ or twisted abelian coefficients
- ▶ **output:** ‘global observables’ $\int_X A$ by gluing (colimit/coend),
e.g. $A \equiv \mathbb{Z}$ produces singular chains;
associative $A \rightsquigarrow \int_{S^1} A = HC(A)$, Hochschild chains.
Goresky–MacPherson’s intersection homology is another special case.

FH 'composes in the opposite way':

- ▶ FFTs: gluing
- ▶ FH: 'merging' or 'collapsing'; morphisms prescribed by cutting

More concretely:

- ▶ FFTs: composition of maps
- ▶ FH: tensor/monoidal products

No wonder:

- ▶ FFTs: time evolution of **states**
- ▶ FH: translates gluing on the underlying spacetime to the gluing of **observables**

Calaque–Scheimbauer, goes back to Lurie:
FFTs from FH: **framed, fully extended**

- ▶ **input:** an E_n -algebra \simeq a framed n -disk algebra
(This is generic by a result of Ayala et al.: any FH theory on framed n -manifolds is uniquely determined by an input E_n -algebra.)
- ▶ **output:** f.e. FFT with values in $\text{Mor}_n(\mathcal{T})$, the/a **Morita category of E_n -algebras** (in some target s-m ∞ -category \mathcal{T})

How?

Take framed collar neighbourhoods, evaluate FH, use Lurie's result that

E_k -algebras \simeq locally constant factorisation algebras on \mathbb{R}^k

to land in $\text{Mor}_n(\mathcal{T})$.

Indeed, in $\text{Mor}_n(\mathcal{T})$:

0: E_n -alg

1: E_{n-1} -alg with compatible E_n -actions (source and target)

2: E_{n-2} -alg with compatible E_{n-1} -actions (with compatible actions on these themselves), etc.

Composition: Merge by tensoring:

$${}_A M_B, {}_B N_C \rightsquigarrow {}_A (M \otimes_B N)_C$$

In terms of FAs, **pushforwards along collapse maps**

FAs more general than FH:

Input algebra A for FH \rightsquigarrow l.c. FA F from local data
 $\rightsquigarrow \int_X A = F(X)$

E.g. in Lurie's theorem:

$F(\mathbb{R}^k)$ gives back the E_k -algebra

or rather its underlying object:

$$F(X) = p_* F \text{ where } p: X \rightarrow *$$

Simplification to E_k -algs possible due to topological reasons

Physically too restrictive: only framed theories allowed

Need: new target Morita categories of FAs beyond framed

Hence **first need**: FAs sensitive to geometric structure

Today: only **tangential** structure, but very general:

- ▶ **Stratified spaces** (boundaries, corners, defects) allowed
- ▶ **Stratified tangential structures** allowed:

$$\text{any } \infty\text{-category } \mathcal{B} \rightarrow \mathcal{V}^{\text{inj}}$$

where \mathcal{V}^{inj} is *BGL* with injections allowed

(Stratified FAs needed for target, even if 'not' for input!)

A (conically-smooth) stratified space X :

continuous $X \rightarrow P$, $P =$ stratifying poset

& useful smoothness properties.

$\Pi_\infty X \rightsquigarrow \mathbf{Ex}(X)$: the **exit path ∞ -category**
(Lurie–MacPherson–Ayala–Francis–Rozenblyum)

‘Bundles’ \rightsquigarrow functors out of $\mathbf{Ex}(X)$

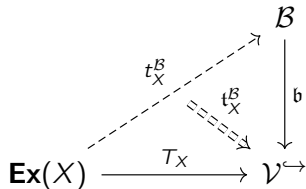
$$T_X: \mathbf{Ex}(X) \rightarrow \mathcal{V}^{\text{inj}}$$

Interaction among strata **over links**:

(path space from X_p to X_q) $\simeq L_{pq}$

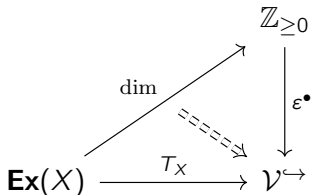
T_X gives bundle map over L_{pq}

\mathcal{B} -reductions:



$$\mathcal{B}\text{-red}(X) = \text{Map}_{/\mathcal{V}^{\hookrightarrow}}(\mathbf{Ex}(X), \mathcal{B})$$

Variframings:

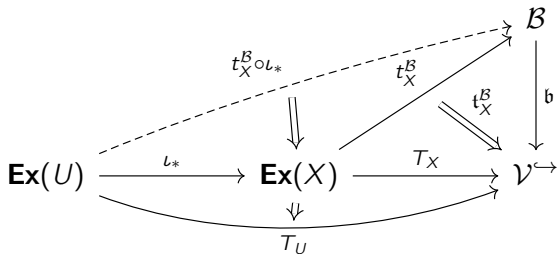


\mathcal{B} -red: (pre)sheaf on stratified spaces and open embeddings:

$$\iota: U \hookrightarrow X$$

\rightsquigarrow

$$\iota^*: \mathcal{B}\text{-red}(X) \rightarrow \mathcal{B}\text{-red}(U)$$



We are led to the **space of \mathcal{B} -open structures** on ι :

$$\mathcal{B}\text{-open}_X(\iota) = \mathcal{M}\text{ap}_{/\iota^* T_X^{\mathcal{B}}}^{\sim}(*, \mathcal{B}\text{-red}(U))$$

and we write

$$T_{\iota}^{\mathcal{B}} \in \mathcal{B}\text{-open}_X(\iota)$$

Smooth sanity check via HTT 5.5.5.12

Recall

pFA on X with coefficients in s-m ∞ -category \mathcal{T} :

algebra over \mathcal{O}_X :

objects: opens of X

k -multihoms: $\mathcal{O}_X(U_1, \dots, U_n | V) = \begin{cases} *, & \text{if } \coprod U_i \subseteq V \\ \emptyset, & \text{else} \end{cases}$

factorising precosheaf:

$$(\mathcal{T}, \otimes) \rightsquigarrow \underline{\mathcal{I}}(a_1, \dots, a_n | b) := \mathcal{T}(a_1 \otimes \dots \otimes a_n, b)$$

Two natural generalisations:

$$\mathcal{O}_X \rightsquigarrow \widehat{\mathcal{O}}_X^{\mathcal{B}} \text{ and } \mathcal{O}_X^{\mathcal{B}}$$

objects:

$$\coprod_{\iota: U \rightarrow X} \mathcal{B}\text{-open}_X(\iota)$$

multihoms in $\widehat{\mathcal{O}}_X^{\mathcal{B}}$:

$$\widehat{\mathcal{O}}_X^{\mathcal{B}}(\alpha_1, \dots, \alpha_k | \beta) = \begin{cases} \prod \widehat{X}^{\mathcal{B}}(\alpha_i | \beta), & \text{if } \coprod U_{\alpha_i} \subseteq U_{\beta} \\ \emptyset, & \text{else} \end{cases}$$

where

$$\widehat{X}^{\mathcal{B}}(\alpha | \beta) = \mathcal{B}\text{-red}_{\alpha}(T_{\alpha}^{\mathcal{B}}, \iota_{\alpha\beta}^* T_{\beta}^{\mathcal{B}}).$$

multihoms in $\mathbb{O}_X^{\mathcal{B}}$:

$$\mathbb{O}_X^{\mathcal{B}}(\alpha_1, \dots, \alpha_k | \beta) = \begin{cases} \prod X^{\mathcal{B}}(\alpha_i | \beta), & \text{if } \coprod U_{\alpha_i} \subseteq U_{\beta} \\ \emptyset, & \text{else} \end{cases}$$

where

$$X^{\mathcal{B}}(\alpha | \beta) = \mathcal{B}\text{-red}_{\alpha}(T_{\alpha}^{\mathcal{B}}, \alpha^* T_{\beta}^{\mathcal{B}}) / \epsilon_{\alpha\beta}$$

where

$$\epsilon_{\alpha\beta}: T_{\alpha}^{\mathcal{B}} \xrightarrow{\sim} \iota_{\alpha\beta}^* T_{\beta}^{\mathcal{B}}$$

is the ‘canonical equivalence’ through X .

Such inclusions of \mathcal{B} -opens: **ambient-compatible**.

Invariant reformulation:

$$\mathcal{B}\text{-red}_\alpha(T_\alpha^\mathcal{B}, \alpha^* T_\beta^\mathcal{B}) / \epsilon_{\alpha\beta} \simeq \mathcal{B}\text{-open}_X(\alpha)(T_\alpha^\mathcal{B}, \iota_{\alpha\beta}^* T_\beta^\mathcal{B})$$

so this is the 'over- ∞ -category over $(X, T_X^\mathcal{B})$ ' approach.

Without this, we cannot decompose structure maps.

Given nested \mathcal{B} -opens $U_\alpha \hookrightarrow U_\beta \hookrightarrow U_\gamma$,

$$\widehat{\circledast}: \widehat{X}^{\mathcal{B}}(U_\alpha | U_\beta) \times \widehat{X}^{\mathcal{B}}(U_\beta | U_\gamma) \rightarrow \widehat{X}^{\mathcal{B}}(U_\alpha | U_\gamma)$$

in the obvious way.

Lemma

Multiplication descends to the hatless version.

Thus:

$$\widehat{\circledast}: \widehat{\mathcal{O}}_X^{\mathcal{B}}(\boldsymbol{\alpha} | \gamma) \times \widehat{\mathcal{O}}_X^{\mathcal{B}}(\boldsymbol{\beta}_1 | \alpha_1) \times \cdots \times \widehat{\mathcal{O}}_X^{\mathcal{B}}(\boldsymbol{\beta}_k | \alpha_k) \rightarrow \widehat{\mathcal{O}}_X^{\mathcal{B}}(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k | \gamma)$$

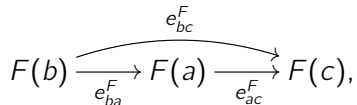
with finite families of nested \mathcal{B} -opens,
as well as the hatless version.

Definition

An $\widehat{\mathcal{O}}_X^{\mathcal{B}}$ -algebra: \mathcal{B} -pFA.

An $\mathcal{O}_X^{\mathcal{B}}$ -algebra: **ambient** \mathcal{B} -pFA.

Decomposition:

$$F(b) \xrightarrow[e_{ba}^F]{} F(a) \xrightarrow[e_{ac}^F]{} F(c),$$


always h-commutes if F is ambient, otherwise not.

Example

$\underline{S^1} = (\mathcal{V}^{\text{inj}} \times S^1)$ -pFAs on $*$.

Generally: **spaces** of structure maps between observables at nested \mathcal{B} -opens:

$$\mathcal{X}^{\mathcal{B}} \left(\coprod \alpha_i \mid \beta \right) \rightarrow \mathcal{T} \left(\bigotimes F(\alpha_i), F(\beta) \right)$$

Ambient \Rightarrow **connected**.

Towards \mathcal{B} -FAs

Can adapt Čech:

$$\check{C}_k^F(\mathbf{U}) := \check{C}_k(\mathbf{U}; F) := \coprod_{A \in PU^{k+1}} \bigotimes_{\alpha \in A} F \left(\bigcap_{\alpha_j \in \alpha} \alpha_j \right)$$

comes with

$$(-)^F : \mathbb{O}_X^{\mathcal{B}}(\cap \alpha \mid \cap \hat{\alpha}^I) \rightarrow \mathcal{T}(F(\cap \alpha), F(\cap \hat{\alpha}^I))$$

instead of just single such maps $F(\cap \alpha) \rightarrow F(\cap \hat{\alpha}^I)$.

In fact, we have maps

$$\mathbb{O}_X^{\mathcal{B}}(\check{C}_n | \check{C}_m) \rightarrow \mathcal{T}(\check{C}_n^F, \check{C}_m^F)$$

where

$$\mathbb{O}_X^{\mathcal{B}}(\check{C}_n | \check{C}_m) := \coprod_{\rho \in \Delta([m], [n])} \prod_{A \in PU^{n+1}} \prod_{\alpha \in A} \mathbb{O}_X^{\mathcal{B}}(\cap \alpha | \cap \rho^* \alpha).$$

Organised by an ∞ -category $\Delta_X^{\mathcal{B}}(\mathbf{U})$

objects: same as Δ

morphisms: $\Delta_X^{\mathcal{B}}(\mathbf{U}) = \mathbb{O}_X^{\mathcal{B}}(\check{C}_m | \check{C}_n)$

Definition

The **ambient \mathcal{B} -Čech complex**:

$$\check{\mathcal{C}}_{\bullet}(\mathbf{U}; F) \in \text{pSh}(\Delta_X^{\mathcal{B}}; \mathcal{T})$$

Proposition (sanity check on 0-truncations)

$$|\Delta_X^{\mathcal{B}}| \simeq \Delta$$

$$|\check{\mathcal{C}}_{\bullet}^F| \simeq \check{\mathcal{C}}_{\bullet}^{|F|}$$

Possibly noncontractible choices in the non-ambient version.

(Weiss/factorising) cosheaf condition via

$$e_k^F: \check{C}_k^F \rightarrow F(U)$$

and

Proposition

$e_{\bullet}^F: \check{C}_{\bullet}^F \rightarrow F(U)$ lifts to a cocone, natural in U , \mathbf{U} .

Local constancy: analogous

$\text{Mor}_n^{\mathcal{B}}(\mathcal{T})$

I.c. \mathcal{B} -FAs on \mathbb{R}^n with flag-type defects

objects: such FAs on \mathbb{R}^n

1-morphisms: with hyperplane defect

2-morphisms: hyperplane defect with codim-2 plane defect, etc.

Compositions: pushforwards along collapse maps (still possible!)
Specific to AFR's constructible tangent bundle

Novel phenomena, questions

No additivity, even with variframings, already on \mathbb{R}^3 with a line defect.

*Essentially because $\pi_4(S^2) \neq *$.*

Invariant reason: structure on **links** does not decompose additively.

Q: To what extent is this an issue?

Poisson: Conjecture/Ansatz: $P_0(\text{Mor}^{\mathcal{B}})$ is enough.

Physical sanity check (scalar field theory: joint with N. Capacci) depends on: The Safronov(–Melani) strict Poisson centre is the categorical centre (à la Lurie).

This is claimed by Safronov.

Q for quantisation: Is the Beilinson–Drinfeld operad additive?
What is the *BD*-centre?

Nontopological geometric structures: our *ambient* \mathcal{B} -FA theory should lift to Grady–Pavlov’s classifying space approach to such geometric structures.

Q/C: The Grady–Pavlov geometric cobordism hypothesis classifies such structured FAs.

\mathcal{B} -Čech nerve theorem? Stratified Artin–Mazur?

The $\widehat{\mathcal{O}}/\mathcal{O}$ distinction like big/little étale site on $(X, T_X^{\mathcal{B}})$.

(a-) \mathcal{B} -FA $(X, T_X^{\mathcal{B}})$ is the (small) factorising cosheaf topos of $(X, T_X^{\mathcal{B}})$.

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Thank you for listening!

References

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