Geometric factorisation algebras and Morita theory

Ödül Tetik

Universität Zürich

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Defects and Symmetry

King's College London

Duality: factorisation homology $\leftrightarrow$ functorial field theories?

FH: homology theories at chain level:

- **input**: disk-algebra A fitting the geometry of spacetime X, e.g. a framed n-disk-algebra for framed n-manifold X
  Nonphysical: $A \equiv \mathbb{Z}$ or twisted abelian coefficients

- **output**: 'global observables' $\int_X A$ by gluing (colimit/coend), e.g. $A \equiv \mathbb{Z}$ produces singular chains;
  associative $A \rightsquigarrow \int_{S^1} A = HC(A)$, Hochschild chains.
  Goresky–MacPherson's intersection homology is another special case.

FH ‘composes in the opposite way’:

- FFTs: gluing
- FH: ‘merging’ or ‘collapsing’; morphisms prescribed by cutting

More concretely:

- FFTs: composition of maps
- FH: tensor/monoidal products

No wonder:

- FFTs: time evolution of **states**
- FH: translates gluing on the underlying spacetime to the gluing of **observables**

Calaque–Scheimbauer, goes back to Lurie:
FFTs from FH: **framed, fully extended**

- **input**: an E_n-algebra $\simeq$ a framed n-disk algebra
  (This is generic by a result of Ayala et al.: any FH theory on framed n-manifolds is uniquely determined by an input E_n-algebra.)

- **output**: f.e. FFT with values in $\mathrm{Mor}_n(\mathcal{T})$, the/a **Morita category of E_n-algebras** (in some target s-m ∞-category $\mathcal{T}$)

How?

Take framed collar neighbourhoods, evaluate FH, use Lurie's result that

$$E_k\text{-algebras} \simeq \text{locally constant factorisation algebras on } \mathbb{R}^k$$

to land in $\mathrm{Mor}_n(\mathcal{T})$.

Indeed, in $\mathrm{Mor}_n(\mathcal{T})$:

0: E_n-alg

1: E_{n-1}-alg with compatible E_n-actions (source and target)

2: E_{n-2}-alg with compatible E_{n-1}-actions (with compatible actions on these themselves), etc.

Composition: Merge by tensoring:

$$ {}_A M_B, \quad {}_B N_C \quad \rightsquigarrow \quad {}_A(M \otimes_B N)_C $$

In terms of FAs, **pushforwards along collapse maps**

FAs more general than FH:

$$\text{Input algebra } A \text{ for FH} \rightsquigarrow \text{l.c. FA } F \text{ from local data}$$
$$\rightsquigarrow \int_X A = F(X)$$

E.g. in Lurie's theorem:

$$F(\mathbb{R}^k) \text{ gives back the } E_k\text{-algebra}$$

or rather its underlying object:

$$F(X) = p_* F \text{ where } p\colon X \to *$$

Simplification to E_k-algs possible due to topological reasons

Physically too restrictive: only framed theories allowed

Need: new target Morita categories of FAs beyond framed

Hence **first need**: FAs sensitive to geometric structure

Today: only **tangential** structure, but very general:

- **Stratified spaces** (boundaries, corners, defects) allowed
- **Stratified tangential structures** allowed:

$$\boxed{\text{any } \infty\text{-category } \mathcal{B} \to \mathcal{V}^{\mathrm{inj}}}$$

where $\mathcal{V}^{\mathrm{inj}}$ is $B\mathrm{GL}$ with injections allowed

(Stratified FAs needed for target, even if ‘not’ for input!)

A (conically-smooth) stratified space X:

$$\text{continuous } X \to P,\ P = \text{stratifying poset}$$

& useful smoothness properties.

$\Pi_\infty X \rightsquigarrow \mathbf{Ex}(X)$: the **exit path ∞-category**
(Lurie–MacPherson–Ayala–Francis–Rozenblyum)

'Bundles' $\rightsquigarrow$ functors out of $\mathbf{Ex}(X)$

$\boxed{T_X \colon \mathbf{Ex}(X) \to \mathcal{V}^{\mathrm{inj}}}$

Interaction among strata **over links**:

(path space from X_p to X_q) $\simeq L_{pq}$

T_X gives bundle map over L_{pq}

$\mathcal{B}$**-reductions**:

$$\begin{tikzcd}
& \mathcal{B} \arrow[d, "\mathfrak{b}"] \\
\mathbf{Ex}(X) \arrow[ur, dashed, "t_X^{\mathcal{B}}"] \arrow[r, "T_X"] \arrow[r, phantom, shift left=3, "\mathfrak{t}_X^{\mathcal{B}}"] & \mathcal{V}^{\hookrightarrow}
\end{tikzcd}$$

$$\mathcal{B}\text{-red}(X) = \mathcal{M}\mathrm{ap}_{/\mathcal{V}^{\hookrightarrow}}(\mathbf{Ex}(X), \mathcal{B})$$

Variframings:

$$\begin{tikzcd}
& \mathbb{Z}_{\geq 0} \arrow[d, "\varepsilon^{\bullet}"] \\
\mathbf{Ex}(X) \arrow[ur, "\dim"] \arrow[r, "T_X"] & \mathcal{V}^{\hookrightarrow}
\end{tikzcd}$$

$\mathcal{B}$-red: (pre)sheaf on stratified spaces and open embeddings:

$$\iota\colon U \hookrightarrow X$$

$$\rightsquigarrow$$

$$\iota^*\colon \mathcal{B}\text{-red}(X) \to \mathcal{B}\text{-red}(U)$$

$$\begin{tikzcd}
 & & \mathcal{B} \arrow[d, "\mathfrak{b}"] \\
\mathbf{Ex}(U) \arrow[urr, dashed, "t_X^{\mathcal{B}} \circ \iota_*"] \arrow[r, "\iota_*"] \arrow[rr, bend right, "T_U"'] & \mathbf{Ex}(X) \arrow[ur, "t_X^{\mathcal{B}}"] \arrow[r, "T_X"] \arrow[u, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & \mathcal{V}^{\hookrightarrow} \arrow[u, phantom, "\Downarrow \mathfrak{t}_X^{\mathcal{B}}"]
\end{tikzcd}$$

We are led to the **space of $\mathcal{B}$-open structures** on ι:

$$\mathcal{B}\text{-open}_X(\iota) = \mathcal{M}\mathrm{ap}^{\sim}_{/\iota^* T_X^{\mathcal{B}}}(*, \mathcal{B}\text{-red}(U))$$

and we write

$$T_\iota^{\mathcal{B}} \in \mathcal{B}\text{-open}_X(\iota)$$

Smooth sanity check via HTT 5.5.5.12

Recall

pFA on X with coefficients in s-m ∞-category $\mathcal{T}$:

algebra over $\mathcal{O}_X$:

objects: opens of X

k-**multihoms**: $\mathcal{O}_X(U_1, \ldots, U_n \,|\, V) = \begin{cases} *, & \text{if } \coprod U_i \subseteq V \\ \emptyset, & \text{else} \end{cases}$

factorising precosheaf:

$$(\mathcal{T}, \otimes) \rightsquigarrow \underline{\mathcal{T}}(a_1, \ldots, a_n \,|\, b) := \mathcal{T}(a_1 \otimes \cdots \otimes a_n, b)$$

Two natural generalisations:

$$\mathcal{O}_X \rightsquigarrow \widehat{\mathbb{O}}_X^{\mathcal{B}} \text{ and } \mathbb{O}_X^{\mathcal{B}}$$

objects:

$$\coprod_{\iota\colon U \hookrightarrow X} \mathcal{B}\text{-open}_X(\iota)$$

multihoms in $\widehat{\mathbb{O}}_X^{\mathcal{B}}$:

$$\widehat{\mathbb{O}}_X^{\mathcal{B}}(\alpha_1, \ldots, \alpha_k \,|\, \beta) = \begin{cases} \prod \widehat{X}^{\mathcal{B}}(\alpha_i \,|\, \beta), & \text{if } \coprod U_{\alpha_i} \subseteq U_\beta \\ \emptyset, & \text{else} \end{cases}$$

where

$$\widehat{X}^{\mathcal{B}}(\alpha \,|\, \beta) = \mathcal{B}\text{-red}_\alpha(T_\alpha^{\mathcal{B}}, \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}}).$$

multihoms in $\mathbb{O}_X^{\mathcal{B}}$:

$$\mathbb{O}_X^{\mathcal{B}}(\alpha_1, \dots, \alpha_k \,|\, \beta) = \begin{cases} \prod X^{\mathcal{B}}(\alpha_i \,|\, \beta), & \text{if } \coprod U_{\alpha_i} \subseteq U_\beta \\ \emptyset, & \text{else} \end{cases}$$

where

$$X^{\mathcal{B}}(\alpha \,|\, \beta) = \mathcal{B}\text{-red}_\alpha(T_\alpha^{\mathcal{B}}, \alpha^* T_\beta^{\mathcal{B}})_{/\mathfrak{e}_{\alpha\beta}}$$

where

$$\mathfrak{e}_{\alpha\beta} \colon \; T_\alpha^{\mathcal{B}} \xrightarrow{\sim} \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}}$$

is the 'canonical equivalence' through X.

Such inclusions of $\mathcal{B}$-opens: **ambient-compatible**.

Invariant reformulation:

$$\mathcal{B}\text{-red}_\alpha(T^{\mathcal{B}}_\alpha, \alpha^* T^{\mathcal{B}}_\beta)_{/\mathfrak{e}_{\alpha\beta}} \simeq \mathcal{B}\text{-open}_X(\alpha)(T^{\mathcal{B}}_\alpha, \iota^*_{\alpha\beta} T^{\mathcal{B}}_\beta)$$

so this is the 'over-∞-category over $(X, T^{\mathcal{B}}_X)$' approach.

Without this, we cannot decompose structure maps.

Given nested $\mathcal{B}$-opens $U_\alpha \hookrightarrow U_\beta \hookrightarrow U_\gamma$,

$$\widehat{\circledast}\colon \widehat{X}^{\mathcal{B}}(U_\alpha \mid U_\beta) \times \widehat{X}^{\mathcal{B}}(U_\beta \mid U_\gamma) \to \widehat{X}^{\mathcal{B}}(U_\alpha \mid U_\gamma)$$

in the obvious way.

Lemma

Multiplication descends to the hatless version.

Thus:

$$\widehat{\circledast}\colon \widehat{\mathbb{O}}^{\mathcal{B}}_X(\boldsymbol{\alpha} \mid \gamma) \times \widehat{\mathbb{O}}^{\mathcal{B}}_X(\boldsymbol{\beta}_1 \mid \alpha_1) \times \cdots \times \widehat{\mathbb{O}}^{\mathcal{B}}_X(\boldsymbol{\beta}_k \mid \alpha_k) \to \widehat{\mathbb{O}}^{\mathcal{B}}_X(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k \mid \gamma)$$

with finite families of nested $\mathcal{B}$-opens,
as well as the hatless version.

Definition

An $\widehat{\mathbb{O}}_X^{\mathcal{B}}$-algebra: $\mathcal{B}$-**pFA**.
An $\mathbb{O}_X^{\mathcal{B}}$-algebra: **ambient** $\mathcal{B}$-**pFA**.

Decomposition:

$$\begin{tikzcd} F(b) \arrow[r, "e^F_{ba}"'] \arrow[rr, bend left, "e^F_{bc}"] & F(a) \arrow[r, "e^F_{ac}"'] & F(c), \end{tikzcd}$$

always h-commutes if F is ambient, otherwise not.

Example

$\underline{S^1} = (\mathcal{V}^{\mathrm{inj}} \times S^1)$-pFAs on $*$.

Generally: **spaces** of structure maps between observables at nested $\mathcal{B}$-opens:

$$X^{\mathcal{B}}\left(\coprod \alpha_i \,|\, \beta\right) \to \mathcal{T}\left(\bigotimes F(\alpha_i), F(\beta)\right)$$

Ambient $\Rightarrow$ **connected**.

Towards $\mathcal{B}$-FAs

Can adapt Čech:

$$\check{\mathcal{C}}_k^F(\mathbf{U}) \coloneqq \check{\mathcal{C}}_k(\mathbf{U}; F) \coloneqq \coprod_{\mathsf{A} \in P\mathsf{U}^{k+1}} \bigotimes_{\boldsymbol{\alpha} \in \mathsf{A}} F\left(\bigcap_{\alpha_i \in \boldsymbol{\alpha}} \alpha_i\right)$$

comes with

$$(-)^F \colon \mathbb{O}_X^{\mathcal{B}}(\cap \boldsymbol{\alpha} \mid \cap \boldsymbol{\alpha}^{\hat{I}}) \to \mathcal{T}(F(\cap \boldsymbol{\alpha}), F(\cap \boldsymbol{\alpha}^{\hat{I}}))$$

instead of just single such maps $F(\cap \boldsymbol{\alpha}) \to F(\cap \boldsymbol{\alpha}^{\hat{I}})$.

In fact, we have maps

$$\boxed{\mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_n \,|\, \check{\mathcal{C}}_m) \to \mathcal{T}(\check{\mathcal{C}}_n^F, \check{\mathcal{C}}_m^F)}$$

where

$$\mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_n \,|\, \check{\mathcal{C}}_m) \coloneqq \coprod_{\rho \in \mathbf{\Delta}([m],[n])} \prod_{\mathsf{A} \in P\mathsf{U}^{n+1}} \prod_{\boldsymbol{\alpha} \in \mathsf{A}} \mathbb{O}_X^{\mathcal{B}}(\cap \boldsymbol{\alpha} \mid \cap \rho^* \boldsymbol{\alpha}).$$

$$\boxed{\text{Organised by an } \infty\text{-category } \mathbf{\Delta}_X^{\mathcal{B}}(\mathbf{U})}$$

objects: same as $\mathbf{\Delta}$

morphisms: $\mathbf{\Delta}_X^{\mathcal{B}}(\mathbf{U}) = \mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_m \,|\, \check{\mathcal{C}}_n)$

Definition

The **ambient $\mathcal{B}$-Čech complex**:

$$\check{\mathcal{C}}_\bullet(\mathbf{U}; F) \in \mathrm{pSh}(\mathbf{\Delta}_X^{\mathcal{B}}; \mathcal{T})$$

Proposition (sanity check on 0-truncations)

$$|\mathbf{\Delta}_X^{\mathcal{B}}| \simeq \mathbf{\Delta}$$

$$|\check{\mathcal{C}}_\bullet^F| \simeq \check{C}_\bullet^{|F|}$$

Possibly noncontractible choices in the non-ambient version.

(Weiss/factorising) cosheaf condition via

$$\mathfrak{e}_k^F : \check{\mathcal{C}}_k^F \to F(U)$$

and

Proposition

$\mathfrak{e}_\bullet^F : \check{\mathcal{C}}_\bullet^F \to F(U)$ *lifts to a cocone, natural in* U, $\mathbf{U}$.

Local constancy: analogous

$\mathrm{Mor}_n^{\mathcal{B}}(\mathcal{T})$

$\boxed{\text{l.c. } \mathcal{B}\text{-FAs on } \mathbb{R}^n \text{ with flag-type defects}}$

objects: such FAs on $\mathbb{R}^n$

1**-morphisms:** with hyperplane defect

2**-morphisms:** hyperplane defect with codim-2 plane defect, etc.

Compositions: pushforwards along collapse maps (still possible!)
Specific to AFR's constructible tangent bundle

Novel phenomena, questions

No additivity, even with variframings, already on $\mathbb{R}^3$ with a line defect.
Essentially because $\pi_4(S^2) \neq *$.
Invariant reason: structure on **links** does not decompose additively.
Q: To what extent is this an issue?

Poisson: Conjecture/Ansatz: $P_0(\mathrm{Mor}^{\mathcal{B}})$ is enough.
Physical sanity check (scalar field theory: joint with N. Capacci) depends on: The Safronov(–Melani) strict Poisson centre is the categorical centre (à la Lurie).
This is claimed by Safronov.
Q for quantisation: Is the Beilinson–Drinfeld operad additive? What is the BD-centre?

Nontopological geometric structures: our *ambient* $\mathcal{B}$-FA theory should lift to Grady–Pavlov's classifying space approach to such geometric structures.
Q/C: The Grady–Pavlov geometric cobordism hypothesis classifies such structured FAs.

$\mathcal{B}$-Čech nerve theorem? Stratified Artin–Mazur?
The $\widehat{\mathbb{O}}/\mathbb{O}$ distinction like big/little étale site on $(X, T_X^{\mathcal{B}})$.

(a-)$\mathcal{B}$-FA$(X, T_X^{\mathcal{B}})$ is the (small) factorising cosheaf topos of $(X, T_X^{\mathcal{B}})$.

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Thank you for listening!

References

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