FIELD THEORY FROM HOMOLOGY VIA POINCARÉ DUALS
$\boxed{\textbf{CHEATSHEET}}$

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For concreteness, first the framed case, but it will be clear from the generality of the treatment that this is no restriction.

Given a framed n-disk algebra, goal: construct fully extended n-dimensional FFT. We do framed first because ideally the construction should reproduce Lurie's suggestion in that case (which it does). Also, one can allow an input disk algebra with defects in the exact same way, and get extended FFTs with defects.

We give a solution up to a conjecture, in the form of an 'integration problem' (of an 'étalé space' and a related much smaller space).

An n-framing on a bordism M, $0 \leq \dim M \leq n$, is a trivialisation of $TM \oplus \varepsilon^{\operatorname{codim} M}$. This is called a *stable n-framing*.

Less strictly, seeing M canonically as a stratified space, we may ask that it possess a *solid n-framing*.

Namely, let $\mathcal{V}^{\hookrightarrow}$ denote the *stratified Grassmannian*, and let

$$*_n \to \mathcal{V}^{\hookrightarrow}$$

be the tangential structure for *n-framings*, which is the map from the point that picks out the object $\mathbb{R}^n \in \mathcal{V}^{\hookrightarrow}$.

It has a *right fibration replacement*

$$\mathbf{s}*_n \simeq \mathcal{V}^{\hookrightarrow}/\mathbb{R}^n \to \mathcal{V}^{\hookrightarrow},$$

the projection from the slice.

A stratified space M together with a (homotopy-)lift of its tangent bundle

$$T_M \colon \mathbf{Ex}(M) \to \mathcal{V}^{\hookrightarrow}$$

to $\mathbf{s}*_n$ is called solidly n-framed.

Explicitly, a solid n-framing on M is the datum of an embedding $T_M \hookrightarrow \varepsilon^n$, whereas stable is the extra datum of a trivialisation of the normal bundle; so

$$\text{stable} \Rightarrow \text{solid}$$

If M is just a bordism, the two structures are equivalent due to the existence of nowhere-zero inward-pointing vector fields.

If M^n is just a smooth manifold (no boundary or defects), then $*_n \equiv \mathbf{s}*_n$ on M.

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Key bug/feature: a $*_n$-disk algebra can only evaluate $*_n$-manifolds, not $\mathbf{s}*_n$-manifolds.

A rapid course on the Grothendieck construction(s).

Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of ∞-categories (only invertible natural transformations).

There is a *universal bundle*

$$\begin{tikzcd} Z \arrow[d, "\text{target}"] \\ \mathcal{C}\mathrm{at}_\infty \end{tikzcd}$$

where Z is the full sub-∞-category

$$\mathcal{C}\mathrm{at}_\infty{}^{[1]}\text{-span}\,\langle[\mathcal{C}/x \to \mathcal{C}]\rangle$$

of the arrow category $\mathcal{C}\mathrm{at}_\infty{}^{[1]}$. This is a coCartesian fibration. We canonically have

$$Z|_{\mathcal{C}} \simeq \mathcal{C}$$

for each fibre.

For any $\mathcal{C}$, one can consider the *'right-internal-Yoneda embedding'*

$$\begin{aligned} \mathbf{R}_{\mathcal{C}} \colon \mathcal{C} &\to \mathcal{C}\mathrm{at}_\infty/\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty \\ x &\mapsto [\mathcal{C}/x \to \mathcal{C}] \mapsto \mathcal{C}/x \\ &\quad \text{(pushforward on morphisms)} \end{aligned}$$

whose image is $Z|_{\mathcal{C}}$, so equivalent to $\mathcal{C}$ itself.

The *covariant Grothendieck construction* at $\mathcal{C}$ can be written concretely as the pullback

$$\int \mathbf{R}_{\mathcal{C}} := \lim\left(\mathcal{C} \xrightarrow{\mathbf{R}} \mathcal{C}\mathrm{at}_\infty \xleftarrow{\mathrm{t}} Z\right)$$

Dually, there is a *'left-internal-Yoneda embedding'*

$$\begin{aligned} \mathbf{L}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} &\to \mathcal{C}\mathrm{at}_\infty/\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty \\ x &\mapsto [x/\mathcal{C} \to \mathcal{C}] \mapsto x/\mathcal{C} \\ [x \to y] &\mapsto [[x/\mathcal{C} \leftarrow y/\mathcal{C}] \Rightarrow \mathcal{C}] \mapsto [x/\mathcal{C} \leftarrow y/\mathcal{C}] \\ &\ \ldots \end{aligned}$$

and a universal cobundle

$$Z^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty{}^{\mathrm{op}}$$

where one can write $Z^{\mathrm{op}} \simeq \mathcal{C}\mathrm{at}_\infty{}^{[1]}\text{-span}\,\langle[x/\mathcal{C} \to \mathcal{C}]\rangle$. Taking the pullback of Z^{op} along $\mathbf{L}_{\mathcal{C}}^{\mathrm{op}}$ is an explicit description of the more familiar (contravariant) Grothendieck construction at $\mathcal{C}$.

Special case: smooth manifolds.

$\int \mathbf{R}_{\mathcal{C}}$ should be thought of as the *'extended' Stiefel bundle* on $\mathcal{C}$.

Indeed, shortening $\mathbf{Ex}(B\mathrm{O}(n)) \simeq \Pi_\infty B\mathrm{O}(n)$ to just $B\mathrm{O}(n)$, consider

$$B\mathrm{O}(n)^{\mathrm{op}} \xrightarrow{\mathbf{L}} \mathcal{C}\mathrm{at}_\infty$$

$$\rightsquigarrow$$

$$\begin{tikzcd} \int \mathbf{L}_{B\mathrm{O}(n)} \arrow[d] \\ B\mathrm{O}(n) \end{tikzcd}$$

As $B\mathrm{O}(n)$ has a single object, $\mathbb{R}^n$, we have

$$\int \mathbf{L}_{B\mathrm{O}(n)} \simeq \int \left[\mathbf{L}_{B\mathrm{O}(n)}\right]\big|_{\mathbb{R}^n} \simeq \mathbb{R}^n/B\mathrm{O}(n) \simeq *$$

since under-∞-groupoids are contractible, and so in fact indeed we have

$$\int \mathbf{L} \simeq E\mathrm{O}(n).$$

The passage from subspaces to frames (Grassmann $\rightsquigarrow$ Stiefel) can be explicitly seen to fall out here by observing

$$\left(\operatorname*{colim}_{N\to\infty} \mathrm{Gr}_n(\mathbb{R}^N)\right)_{/\mathbb{R}^n} \simeq \operatorname*{colim}_{N\to\infty} \left(\mathrm{Gr}_n(\mathbb{R}^N)\right)_{/\mathbb{R}^n} \simeq \operatorname*{colim}_{N\to\infty} V_n(\mathbb{R}^N).$$

Also, we can do the opposite-same with covariant Grothendieck: take $\mathbf{R}\colon B\mathrm{O}(n) \to \mathcal{C}\mathrm{at}_\infty \rightsquigarrow \int^{\mathrm{op}} \mathbf{R} \simeq B\mathrm{O}(n)/\mathbb{R}^n \simeq *$. Oppositeness is immaterial since $B\mathrm{O}(n)$ is an ∞-groupoid.

The stratified Stiefel bundle.

We need to fine-tune the 'internal Grothendieck bundle' $\int \mathbf{R}_{\mathcal{V}^{\hookrightarrow}} \to \mathcal{V}^{\hookrightarrow}$, since it gives 'all-codimension frames'.

Let

$$\boldsymbol{\varepsilon} \subset \mathcal{V}^{\hookrightarrow},$$

the *anti-Grassmannian*, be the (non-full) sub-∞-category whose objects are $\boldsymbol{n}$, $n \geq 0$, and whose morphism spaces are given by

$$\boldsymbol{\varepsilon}(\boldsymbol{n},\boldsymbol{m}) = \begin{cases} *, & n = m \\ \mathcal{V}^{\hookrightarrow}(\boldsymbol{n},\boldsymbol{m}), & n \neq m. \end{cases}$$

When $n < m$, we have $\boldsymbol{\varepsilon}(\boldsymbol{n},\boldsymbol{m}) \simeq V_n(\boldsymbol{m}) \simeq \mathrm{O}(m)/\mathrm{O}(m-n)$.

We will call its 'free Cartesian replacement'

$$\begin{tikzcd} \mathcal{E}\mathcal{V}^{\hookrightarrow} \arrow[d, "\text{source}"] \\ \mathcal{V}^{\hookrightarrow} \end{tikzcd}$$

where

$$\begin{tikzcd} \mathcal{E}\mathcal{V}^{\hookrightarrow} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathcal{V}^{\hookrightarrow})^{[1]} \arrow[d, "\text{target}"] \\ \varepsilon \arrow[r, hook] & \mathcal{V}^{\hookrightarrow} \end{tikzcd}$$

the *stratified Stiefel bundle*. It can be written as a certain restriction of the right-internal Yoneda.

Upon restriction to a fixed dimension n, it recovers the ordinary Stiefel bundle:

$$\begin{tikzcd} \int \mathbf{R} = E\mathrm{O}(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}\mathcal{V}^{\hookrightarrow} \arrow[d] \\ B\mathrm{O}(n) \arrow[r, hook] & \mathcal{V}^{\hookrightarrow} \end{tikzcd}$$

This is not the case when the naïve $\int \mathbf{R}_{\mathcal{V}^{\hookrightarrow}} \to \mathcal{V}^{\hookrightarrow}$ is used.

Some linear algebra.

Fix an inner product on $\boldsymbol{n} := \mathbb{R}^n$, and let $\mathcal{V}^{\twoheadrightarrow}$, the *stratified op/co-Grassmannian*, be the version of $\mathcal{V}^{\hookrightarrow}$ with surjections allowed instead of injections. Will need:

$$\begin{aligned} \flat\colon (\mathcal{V}^{\hookrightarrow})^{\mathrm{op}} &\simeq \mathcal{V}^{\twoheadrightarrow} \\ V &\mapsto V^\vee \\ [V \hookrightarrow W] &\mapsto [W^\vee \twoheadrightarrow V^\vee] \\ &\dots \end{aligned}$$

$$\begin{aligned} \flat\colon (\mathcal{V}^{\hookrightarrow}/\boldsymbol{n})^{\mathrm{op}} &\simeq \boldsymbol{n}^\vee/\mathcal{V}^{\twoheadrightarrow} \\ [V \hookrightarrow \boldsymbol{n}] &\mapsto [\boldsymbol{n}^\vee \twoheadrightarrow V^\vee] \\ [V \hookrightarrow W]/\boldsymbol{n} &\mapsto \boldsymbol{n}^\vee/[W^\vee \twoheadrightarrow V^\vee] \\ &\dots \end{aligned}$$

$$\begin{aligned} \sharp\colon (\boldsymbol{n}/\mathcal{V}^{\twoheadrightarrow})^{\mathrm{op}} &\simeq (\mathcal{V}^{\hookrightarrow}/\boldsymbol{n}^\vee) \\ [\boldsymbol{n} \twoheadrightarrow V] &\mapsto [V^\vee \hookrightarrow \boldsymbol{n}^\vee] \\ \boldsymbol{n}/[V \twoheadrightarrow W] &\mapsto [W^\vee \hookrightarrow V^\vee]/\boldsymbol{n}^\vee \\ &\dots \end{aligned}$$

$$\begin{aligned} \perp\colon \mathcal{V}^{\hookrightarrow}/\boldsymbol{n} &\simeq (\mathcal{V}^{\hookrightarrow}/\boldsymbol{n})^{\mathrm{op}} \\ [V \hookrightarrow \boldsymbol{n}] &\mapsto [V^\perp \hookrightarrow \boldsymbol{n}] \\ [V \hookrightarrow W]/\boldsymbol{n} &\mapsto [W^\perp \hookrightarrow V^\perp]/\boldsymbol{n} \\ &\dots \end{aligned}$$

We will not distinguish a functor and its opposite in notation.

Normal cobundles and Poincaré duals.

From above:

$$\mathbf{s}*_n^{\mathrm{op}} \overset{\flat}{\simeq} \mathbb{R}^n/\mathcal{V}^{\twoheadrightarrow}.$$

The coslice projection to $\mathcal{V}^{\twoheadrightarrow}$ becomes $(\mathbf{s}*_n \to \mathcal{V}^{\hookrightarrow})^{\mathrm{op}}$ under $\flat$ and $\sharp$.

Consider a solid n-framing $T_M^{\mathbf{s}*_n}$ on M, whose lifting edge we will denote by

$$t = t_M^{\mathbf{s}*_n} \colon \mathbf{Ex}(M) \to \mathbf{s}*_n.$$

Let $\mathbf{En}(M) = \mathbf{Ex}(M)^{\mathrm{op}}$, the *enter-path ∞-category.*

Induced cobundle on M:

$$\mathfrak{c}(M) \colon \mathbf{En}(M) \xrightarrow{t} (\mathcal{V}^{\hookrightarrow}/\boldsymbol{n})^{\mathrm{op}} \overset{\perp}{\simeq} \mathcal{V}^{\hookrightarrow}/\boldsymbol{n}$$

and

$$\mathfrak{P}(M) \coloneqq \mathrm{Im}(\mathfrak{c}(M)) \subseteq \mathbf{s}*_n.$$

The $\mathbf{Ex}$ of (the 'stratified frame bundle' of) this cobundle is as follows. Consider

$$\begin{tikzcd} \mathcal{E}\mathbf{s}*_n \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}\mathcal{V}^{\hookrightarrow} \arrow[d] \\ \mathbf{s}*_n \arrow[r] & \mathcal{V}^{\hookrightarrow} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{C}(M) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}\mathbf{s}*_n \arrow[d] \\ \mathbf{En}(M) \arrow[r, "\mathfrak{c}"] & \mathbf{s}*_n \end{tikzcd}$$

and note the canonical map

$$\mathfrak{c} \circ \pi \colon \mathfrak{C}(M) \to \mathfrak{P}(M).$$

We are *almost* seeing two new stratified spaces.

The ∞-category $\mathfrak{C}(M)$ comes with canonical tangential data:

- its *horizontal tangent cobundle*

$$T^h \colon \mathfrak{C}(M) \to \mathbf{En}(M) \xrightarrow{t} \mathbf{s}*_n^{\mathrm{op}} \to \mathcal{V}^{\twoheadrightarrow},$$

- its *vertical tangent bundle*

$$T^v \colon \mathfrak{C}(M) \to \mathbf{En}(M) \xrightarrow{\mathfrak{c}} \mathbf{s}*_n \to \mathcal{V}^{\hookrightarrow},$$

- and its *full tangent bundle:*

$$T \colon \mathfrak{C}(M) \to \mathbf{En}(M) \to \mathbf{s}*_n \xrightarrow{\boxplus} \mathcal{V},$$

where we employed

$$\boxplus \colon \mathbf{s}*_n \xrightarrow{\mathrm{diag}} \mathbf{s}*_n \times \mathbf{s}*_n \xrightarrow{\mathrm{id}\times\perp} \mathbf{s}*_n \times \mathbf{s}*_n^{\mathrm{op}} \to \mathcal{V}^{\hookrightarrow} \times \mathcal{V}^{\twoheadrightarrow} \xrightarrow{\oplus} \mathcal{V}.$$

which by construction factors through $T^v \times T^h$.

Crucially, it canonically satisfies

$$\boxed{T_{\mathfrak{C}} \simeq \varepsilon^n}$$

and so in particular factors through $\mathcal{V}^{\hookrightarrow}$.

Morita- or structured realisations.

We want $\mathbf{C}_M \coloneqq |\mathfrak{C}(M)|_{T\simeq\varepsilon^n}$, a stratified space, to be defined by requiring

(1) $\mathbf{Ex}(\mathbf{C}_M) \simeq \mathfrak{C}(M)$

(2) $\left(\mathbf{Ex}(\mathbf{C}_M) \xrightarrow{T_{\mathbf{C}}} \mathcal{V}^{\hookrightarrow}\right) \simeq \left(\mathbf{Ex}(\mathbf{C}_M) \simeq \mathfrak{C}(M) \xrightarrow{T} \mathcal{V}^{\hookrightarrow}\right)$

and analogously (special case of the above), for $\mathbf{P}_M$, we want

(1) $\mathbf{Ex}(\mathbf{P}_M) \simeq \mathfrak{P}(M)$

(2) $\left(\mathbf{Ex}(\mathbf{P}_M) \xrightarrow{T_{\mathbf{P}}} \mathcal{V}^{\hookrightarrow}\right) \simeq \left(\mathbf{Ex}(\mathbf{P}_M) \xrightarrow{\text{forget}} \mathcal{V}^{\hookrightarrow}\right)$

In general, a stratified space with given tangential data analogously to the above need not exist.

Some simple cases:

(1) If M is closed and $\dim M = n$, then $* \simeq \mathfrak{P}(M) = \{[0 \hookrightarrow \boldsymbol{n}]\}$, so $\mathbf{P}_M = *$, $\mathbf{C}_M = M$.

(2) If M is closed and