Towards Linked Topology

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Of quantities some are discrete, others continuous... Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and place. For the parts of a number have no common boundary at which they join together... A line, on the other hand, is a continuous quantity. For it is possible to find a common boundary at which its parts join together, a point. And for a surface, a line; for the parts of a plane join together at some common boundary. Similarly in the case of a body one could find a common boundarya line or a surfaceat which the parts of the body join together.

Categories
Aristotle

Contents

Abstract

We introduce the notions of linked space, linked quasi-category and linked manifold, which are certain spans of the ordinary versions of the respective objects, and which model stratified spaces of various kinds. We then transfer, in depth 1, certain phenomena and constructions from stratified topology to this setting, such as exit path quasi-categories and the beginnings of a stratified bundle theory. We then discuss and extend the topology underlying a construction of J. Lurie, which associates a functorial field theory to any framed disk algebra, to arbitrary tangential structure, as well as an incorporation of defects.

Original work and self-plagiarism

Most of Section 2.2 and Chapter 3 up to and *ex*cluding Section 3.3 have appeared in my preprint [71]. Parts of Chapter 4, including its main result, have appeared in my preprint [72], but are produced here with an improved presentation and with some new content. Chapter 1 also includes some material adapted from the introductory sections of these two preprints. The rest of the text has not been published before. All unattributed results, excepting some classical ones recalled in the preliminary Chapter 2, are original.

Preface

> KINDNESS, *n.*
> A brief preface to ten volumes of exaction.
>
> *The Devil's Dictionary*
> Ambrose Bierce

The ideas that have made it into this dissertation formed over a number of years I spent working on various problems that are related to one another – it seems reasonable to think – to one extent or another. Its contents, therefore, reflect a skimming and homogenisation that was perhaps unnatural, but necessary to present a coherent story. Moreover, it is a far cry from the original research proposal I made back in 2020, and so I think of it as a prologue to other projects to come. If, along the way, I have made some contribution that is of more general interest than only to myself, then I will count the time spent typing well-spent.

There are barely any advanced prerequisites to prevent an enjoyable reading of the text. I have gathered some preliminary material, mostly on quasi-categories, in Chapter 2, within the main text rather than in an appendix. Its purpose is to recall, in otherwise uncharacteristically terse French style, some well-known definitions and facts used throughout the text, and fix some notation. I have endeavoured to keep it minimal but complete. There are a number of external results cited and used in other chapters, which seemed ill-suited for full recitation in Chapter 2 or elsewhere. In such cases, I have provided precise coordinates. If there are no accompanying remarks, the cited result should apply without modification. In the occasion a work is cited without further coordinates, then none of its results are logically required for the statement being made. For the many results I cite from J. Lurie's *Kerodon* I have opted to give tags, which should be stable over time, but result in an unorthodox citation style, such as '[52, 014H].'

I write 'abelian,' 'cartesian,' 'riemannian,' etc., without capitalising, because these words are adjectives. The royal *we* is used – if the reader will allow a Kantian [sic] mannerism – to refer to any entity capable of using the pronoun *I*. Consequent changes in reading are understood: *we see that* means *I see that*; *we will see that* means *I have seen that*; and so on. I have attempted, at times unsuccesfully, to write variables in italics and constants in roman type. Finally, notation interrupts ordinary language according to the following rule: it comes after the noun that refers to it, and not only after all of its qualifications are listed. That is, I write 'a functor $f\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories' rather than 'a functor of ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$.' This is the way.

In the last few years, I have enjoyed the immense academic freedom granted to me by my advisor, Alberto S. Cattaneo. I thank him for his trust and genial company – mathematical and otherwise. My work and attitude have benefitted from exchanges with many people, among whom I should like to mention Iakovos Androulidakis, Kadri İlker Berktav, Giovanni Canepa, Nicola Capacci, Ivan Contreras, Lennart Döppenschmitt, Marius Furter, Aleksandar Ivanov, Emil Jacobsen, Branko Juran, Artem Kalmykov, Thomas Lehéricy, Philippe Mathieu, Louann Rieger, Pavel Safronov, and Çağrı Sert. Thanks are due as well to Joseph Ayoub, John Francis, and Thomas Willwacher for their generous support. I thank the staff at the institute for their constant help, especially Jessica Bolsinger, Bettina Kurth, Gunnar Lenz, and Carsten Rose; and the staff at Kraftwerk for their impeccable operation and countless doppios. A number of people's influence has seeped through, directly or indirectly, into my work: Dennis Borisov, Jörg Brüdern, Thorsten Hertl, Philipp Kastendieck, Mark Penney, Thomas Schick, Ulrich Stuhler, Peter Teichner; Anne, Carl, Charlotte, Jakob, and Max; Francisco, Luis, Marco, and many more besides; and Nick, Raoul, Yasmin and company – thank you all. Call me if you find a mistake so I can remove your name from this paragraph.

My deepest thanks are due to my family for tolerating me over the years: my late father Erol, my late grandfather İrfan, who used to talk physics with me when I was just old enough to remember, my mother Şermin, my grandmother Ünver; Hans, Mattia, Rahel, Salvi, Samira – and my dearest Tabea: I look up to you. You'll have to continue living with my mistakes.

CHAPTER 1

Introduction

1.1. Good spans of spaces

Many results in stratified topology tend to characterise stratified spaces or stratified maps in terms of strata and their links. These links can be either geometric, like the boundary of a 'regular neighbourhood,' such as the boundary of the blow-up along a singularity, or the sphere bundle of the normal bundle of a submanifold; or they can be 'homotopical' in nature, defined to be path spaces between pairs of strata, or higher-depth analogues of such.

The advantage that such results provide is that the strata and links of a stratified space are *smooth*: they are non-stratified, ordinary spaces connected by maps between them, and one can hope to transfer techniques of ordinary topology to study such systems, and thus obtain results about the original stratified space. For instance, in the context of homotopically stratified spaces à la Quinn [63], Miller showed in [57, Theorem 6.3] that stratified homotopy equivalences between such spaces are exactly those maps which induce weak equivalences (in the ordinary sense) on strata and homotopy-links. This means that the strata and the links determine the stratified homotopy type.

There are similar results in the more recent conically-smooth variety developed by Ayala, Francis, Rozenblyum and Tanaka [9, 6], which is a geometric refinement of the conically-stratified spaces formalised by Lurie in [50, Appendix A], which generalises the pseudo-manifolds of the early Whitney–Thom ([80, 73]) days of the theory.[1] Lemma 3.3.5 of [6] identifies the space of paths between strata in terms of links.

It is also of great interest to obtain similar results for stratified disk algebras, or more generally for factorisation algebras locally constant with respect to some stratification, and indeed this has been achieved in some paradigmatic cases in the conically-smooth context. The prototypical statement is Deligne–Kontsevich's Swiss-Cheese Conjecture (Theorem) ([49]),[2] which, in one formulation, states that a Swiss-Cheese algebra ([79]) in dimensions n, $n-1$ (i.e., on the n-dimensional half-plane) is equivalent to an $\mathbb{E}_n$-algebra A (its restriction to the interior), an $\mathbb{E}_{n-1}$-algebra M (its 'restriction' to the boundary), and a map $A \to \mathrm{HC}(M)$ of $\mathbb{E}_n$-algebras (the action of A on M),

[1] Indeed, Whitney-stratified spaces are conically smooth: see [60]. See [62] for an in-depth treatment of analytic and geometric aspects of stratified space theory, as well as a historical account of the developments in the 20th century. In the present work, we will only focus on stratified *topology*.

[2] See e.g. [23], but also [74] for a proof of the statement we give as well as a historical overview of earlier proofs of the various incarnations of the theorem.

with target the Hochschild cochain object, a model for the centre of M in the sense of [50, §5].[3] Similarly, Ayala–Francis–Tanaka considered algebras on closed intervals and on euclidean space stratified by a distinguished hyperplane in [8, §2.6 resp. §4.3], obtaining similar algebraic characterisations. In the latter case, the action of the n-dimensional bulk algebra A on the d-dimensional hyperplane algebra M is characterised, if we disregard some details concerning tangential structure, by a $(d+1)$-algebra map $\int_{S^{n-d-1}} A \to \mathrm{HC}(M)$, with the link S^{n-d-1} (or rather $S^{n-d-1} \times \mathbf{R}^d$)[4] making a crucial and telling appearance.

In Chapter 3, we propose a construction intended to turn such results around and build stratified spaces directly from strata and links, that is, from collections of smooth spaces that are related by no more than ordinary maps between them. More specifically, we restrict ourselves mostly to depth 1 in this work,[5] where, for every well-behaved span

$$\begin{tikzcd} & L \arrow[dl, two heads, "\pi"'] \arrow[dr, hook, "\iota"] & \\ M & & N \end{tikzcd}$$

of spaces, we construct an ∞-category $\mathcal{EX}$. Here, M and N model two strata, the former lower than the latter, L their link, and $\mathcal{EX}$ the exit path ∞-category à la Lurie–MacPherson–Treumann–Woolf ([76, 82, 50]). The *link maps* π and ι are required to satisfy conditions that combine phenomena in the homotopically stratified as well as in the pseudo-manifold settings. Absent these properties, the construction does not work: when ι is arbitrary, $\mathcal{EX}$ is not even well-defined, and when π is arbitrary (but ι well-behaved), then the construction yields a simplicial set $\mathcal{EX}$ which need not be an ∞-category.

First, the homotopy-link L between two strata M and N in a homotopically stratified set is defined to be the space $L = \mathrm{P}_{M,N}$ of paths that start in M and end in N. Consequently, there is the source evaluation $\pi = \mathrm{ev}_0 \colon \mathrm{P}_{M,N} \to M$, which, in a homotopically stratified set, is required (by Quinn's original definition) to be a fibration. Similarly, we require $\pi \colon L \to M$ to be a fibration.

Second, say the link of a singular submanifold M within a pseudo-manifold $\overline{N}$ (so that $N = \overline{N} \smallsetminus M$) is given by the boundary of its blow-up along M, say by the sphere bundle $\mathbb{S} = \mathbb{S}(\mathrm{N}M)$ of the normal bundle of the submanifold. Then, we may consider the projection $\pi \colon L = \mathbb{S} \to M$ which is certainly a fibration, but then there is also the map $\iota \colon \mathbb{S} \hookrightarrow N$ which is a closed embedding, and in particular a cofibration. Indeed, we require $\iota \colon L \hookrightarrow N$ to be a cofibration. We call a span $\mathfrak{S}$ as above with π a fibration and ι a cofibration a *linked space*. In fact, ι can be simply a continuous injection. The examples that are central to this work involve spans of infinite Grassmannians (Example 3.2.17), and bordisms and defects (submanifolds) in Examples 3.2.15 and 3.2.16.

[3] It is therefore naturally an associative algebra in $\mathbb{E}_{n-1}$-algebras and thus (by Dunn–Lurie additivity [50, 29]) an $\mathbb{E}_n$-algebra, so that we may speak of $\mathbb{E}_n$-maps $A \to \mathrm{HC}(A)$.

[4] We have $\int_{S^{n-d-1}} A = \int_{S^{n-d-1} \times \mathbf{R}^{d+1}} A = \int_{L \times \mathbf{R}} A$ by definition.

[5] We do discuss higher depth; most interestingly, the construction that we are about to describe is iterable – see below.

The construction of the ∞-category $\mathcal{EX} = \mathcal{EX}(\mathfrak{S})$ is based on the following idea. Every point $\ell \in L$ can identified with a 1-morphism of type $M \ni \pi(\ell) \to \iota(\ell) \in N$. Relaxing this slightly, we may consider any path γ in N that starts in $\iota(L)$, and take it as a 1-morphism of type $\pi(\iota^{-1}\gamma_0) \to \gamma_1$. The constant loop inclusion $L \hookrightarrow \mathrm{P}(L) \hookrightarrow \mathrm{P}(N)$ recovers the idea of taking the points of L as 1-morphisms. The simplicial set $\mathcal{EX}$ has objects only the points of M and N, and 1-morphism the paths in M and N and moreover, separately, all paths in N that start in $\iota(L)$. Such paths, the 'exit paths,' are clearly non-invertible since there are, by construction, no morphisms from N to M.

Constructing simplices of higher dimensions in such a way that $\mathcal{EX}$ becomes an ∞-category is most of all a 'combinatorial' challenge. We define the n-simplices to be the n-simplices (of M and N together with those) in N such that, most importantly, its restriction along $\Delta^{\{0,1,\ldots,e-1\}} \hookrightarrow \Delta^n$ for some $n+1 \geq e \geq 1$ lies wholly within $\iota(L)$. We call e the *exit index* of the simplex in question. The idea is that, in depth 1, this index, together with the underlying simplex in N, determines how and where the latter comes into contact with M and N. For instance, if a 2-simplex is (witnesses) a composition of a path γ in M and an exit path δ, then it will be underlied by a 2-simplex $\Gamma\colon \Delta^2 \to N$ such that its 01-edge $\Gamma|_{01} = d_2(\Gamma)$, i.e., its restriction along $\Delta^{\{0,1\}} \hookrightarrow \Delta^2$ is in $\iota(L)$, and such that $\pi(\Gamma|_{01}(0)) = \gamma(0)$,[6] $\pi(\Gamma|_{01}(1)) = \gamma(1)$, and of course $\delta(0) = \Gamma|_{01}(1)$:

$$\begin{tikzcd}
& \ast \\
\ast \arrow[r, "\Gamma|_{01}"'] \arrow[ur, dotted] \arrow[d, dashed, "\pi"] & \ast \arrow[u, "\delta"'] \arrow[d, dashed, "\pi"] \\
\ast \arrow[r, "\gamma"] & \ast
\end{tikzcd}$$

Here, the bottom row is within M, and the triangle depicts the 2-simplex Γ of N which underlies the composition 2-simplex, which is depicted by the whole picture. The 01-edge of this 2-simplex is in fact γ by construction, and its 12-edge is the exit path δ, which is 1-morphism with source $\pi(\delta_0) = \gamma(1) \in M$. This particular composition is a 2-simplex of *exit index* 2 in our convention, owing to the fact that the top-most vertex, which is number 2, is the first one that has exited into the 'higher stratum' N. In order to compose an exit (1-)path γ from $\pi(\gamma_0)$ to $\gamma(1)$ with a path δ in N starting at $\gamma(1)$, we introduce exit 2-paths of *index* 1 into $\mathcal{EX}$, which can be depicted as follows:

$$\begin{tikzcd}
\ast \arrow[r, "\delta"] & \ast \\
\ast \arrow[u, "\gamma"] \arrow[ur, dotted] \arrow[d, dashed, "\pi"] & \\
\ast &
\end{tikzcd}$$

[6]We suppress ι^{-1}.

Here the exit index is 1 because already the vertex 1 has exited into N. There are similar pictures in higher dimensions, where in dimension n there are n different possibilities for the exit index, giving n different classes of n-simplices, which interact with each other appropriately upon application of (the appropriately defined) face and degeneracy maps.

The combinatorial nature of the construction above lets us apply it in greater generality than with input topological spaces only. In fact, we prove the following:

THEOREM (3.2.11). *Let $\mathcal{M}$, $\mathcal{L}$, $\mathcal{N}$ be ∞-categories, $\pi\colon \mathcal{L} \to \mathcal{M}$ a right fibration, and $\iota\colon \mathcal{L} \to \mathcal{N}$ a cofibration. Then $\mathcal{E}\mathcal{X}\left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ is an ∞-category.*

A consequence of this degree of generality is that the construction is iterable: given a linked space (or ∞-category) $\mathfrak{S}$, we can build its exit path ∞-category, and use that as input for a span e.g. of type $\mathfrak{S}' = (\mathcal{E}\mathcal{X}(\mathfrak{S}) \leftarrow \mathcal{L}' \to \mathcal{N}')$ to construct $\mathcal{E}\mathcal{X}(\mathfrak{S}')$, and so on, which will model exit paths in higher depth. This is the topic of Section 3.5; the rest of this work is independent of such conjectural thoughts.

We call spans of ∞-categories of the type above *linked ∞-categories*. Given the ubiquity of right fibrations of ∞-categories in view of their equivalence to space-valued presheaves (via (un)straightening – see [51, §3] or [17, §5] for textbook accounts), the result suggests a practical approach to implement ideas of stratified topology in many different contexts.

Before we explain why this is a *good* construction so that $\mathcal{E}\mathcal{X}$ does indeed behave like the exit path ∞-category of a stratified space, let us mention that philosophically similar ideas have already appeared in the literature. Douteau, in [25], gives a Quillen equivalence between a certain model category of stratified spaces and a model category of diagrams of simplicial sets indexed over (non-degenerate sequences in) posets. Because the bulk of this dissertation is concerned with stratified bundle theory in disguise, the construction of the stratified *simplicial set* (which is then realised to a stratified topological space) associated with a diagram of simplicial sets is too unwieldy for our purposes. It does not yield ∞-categories in general (see [26, Recollection 2.53 ff.]),[7] which we certainly need, and is defined as a certain colimit (computing a certain left Kan extension – see [26, Recollection 2.37]), which makes it rather impractical for some of our necessarily hands-on constructions. This is not unexpected, since the construction is completely unburdened by topological assumptions on the link maps such as the ones we have. However, the work of Douteau et al. contains a wealth of ideas that may be useful in pushing the idea of $\mathcal{E}\mathcal{X}$ further, especially to higher depth without relying on the iteratibility of the construction. We leave such questions to future work.

Now, the following results show that $\mathcal{E}\mathcal{X}$ behaves as one would wish it to.

[7]It is however the case that the exit path ∞-categories associated with conically-smoothly, conically, and homotopically stratified spaces (sets) are all ∞-categories ([50, 59, 6]).

THEOREM (3.3.1). *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ be a linked ∞-category, and $p \in \mathcal{M}$ and $q \in \mathcal{N}$ points in the two strata. We then have an equivalence*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \mathcal{P}_{\mathcal{L}_p, q}$$

between the morphism space in $\mathcal{EX}$ from p to q and that of paths in $\mathcal{N}$ that start in the embedded fibre $\iota(\mathcal{L}_p)$, where $\mathcal{L}_p = \{p\} \times_{\mathcal{M}} \mathcal{L}$, and end in q.

This is a pointwise statement. We globalise it in Section 3.4 for linked spaces and prove

THEOREM (3.4.1). *Let $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked space. Then*

$$L \simeq (M \downarrow N).$$

Here, $(M \downarrow N) = M \times_{\mathcal{EX}^{\{0\}}} \mathcal{EX}^{\Delta[1]} \times_{\mathcal{EX}^{\{1\}}} N$ is the oriented fibre product of $M, N \hookrightarrow \mathcal{EX}$. We should note that their ∞-categorical homotopy fibre product (in the sense of [52, 032Z]) is empty, because, as we already noted, it is clear from the construction that $\mathcal{EX}$ contains no isomorphisms from M to N. The result above is not (and should not be) true for linked ∞-categories in general; instead, we expect [6, Lemma 3.3.5] to hold mutatis mutandis.

Believing it to be too soon, and because it is unnecessary for our purposes in this work, we will not be concerned with setting up a *model* category of linked spaces (or of linked ∞-categories). Hoewever, we will propose a notion of a map of linked spaces in Section 7.1 that recovers and extends ordinary stratified maps, and, at least in depth 1, propose a *linked realisation*, which is sometimes conically smooth, of a *constructible* linked *manifold* in Definition 7.1.14, by repurposing an idea from [6]. A better name might have been 'stratified realisation,' but this might lead to confusion with ideas of, say [24, 25] or [6].

1.2. The tangential theory

For the purpose of constructing a theory of factorisation homology that can take as input any (∞, n)-category and evaluate it on appropriate variframed stratified spaces, Ayala–Francis–Rozenblyum defined in [7] the 'fibrewise constructible tangent bundle' $\mathrm{T}^{\mathrm{fib}}$ of a (conically-smooth) stratified space, which intrinsically depended on their earlier work, in part with Tanaka, on the general theory of conically-smooth stratified spaces ([9, 6]). The functor $\mathrm{T}^{\mathrm{fib}}$ (or rather its nonrelative special case to which we restrict our attention) on a stratified space X is given in the form of a classifying map

$$\mathrm{T}^{\mathrm{fib}} \colon \mathrm{Exit}(X) \to \mathcal{V}^{\mathrm{inj}},$$

whose domain is their version of the exit path ∞-category. It is equivalent to the model of Lurie–MacPherson by a result of [6]. The target $\mathcal{V}^{\mathrm{inj}}$ is what we will call the 'stratified Grassmannian,' an ∞-category that assembles the fundamental ∞-groupoids of the ordinary infinite Grassmannians of all ranks, and adds non-invertible paths between them that increase rank.

By exodromy, such a functor classifies a constructible sheaf on X, which may be interpreted as the sheaf of sections of the 'tangent bundle' – although, to our knowledge, no étalé space of this sheaf has been discussed in the literature. Combining the bundle span featuring in Construction 6.1.5 with the linked realisation of Definition 7.1.14 provides such a space in depth 1, and Chapter 5 is mainly concerned with placing its quasi-categorical classifier within the model of $\mathcal{V}^{\mathrm{inj}}$ we give in Chapter 4.

In keeping with the theme of the previous section, we present a construction of a quasi-categorical variant, $\mathcal{V}^{\hookrightarrow}$, of $\mathcal{V}^{\mathrm{inj}}$ that does not rely on stratified space theory. Namely, Section 4.1 constructs a topological monoid whose operation is given by direct-summing vector spaces. To this end, we circumvent the more systematic treatment of spectra with $\mathbb{E}_\infty$-structure ([50]) or ultra-commutativity ([68]) by adding some redundancy that achieves on-the-nose associativity. In order to develop a real K-theory (spectrum) for linked/stratified spaces, one should pursue a different treatment, but our construction may be informative with respect to its zeroeth space.

Even though the direct (Whitney) sum operation on vector bundles is commutative up to canonical isomorphism, the corresponding operation on classifying spaces is only homotopy-commutative. In fact, the maps

$$\oplus\colon B\mathrm{O}(m) \times B\mathrm{O}(n) \to B\mathrm{O}(n+m)$$

induced by direct-summing rank-m and rank-n vector subspaces of $\mathbf{R}^\infty$ is also not associative, but only so up to (contractible) homotopy. We give a straightforward strictification of $(\coprod_{k\geq 0} B\mathrm{O}(k), \oplus)$, obtaining a topological monoid

$$(B\mathrm{O}^\infty_{\amalg}, \oplus).$$

We note in passing that there is a very non-canonical homeomorphism

$$B\mathrm{O}^\infty_{\amalg} \cong * \amalg \mathbf{Z}_+ \times \coprod_{k\geq 1} B\mathrm{O}(k),$$

where the extra factors on the right are a result of the strictification.

Now, finally able to follow the idea of [7, Remark 2.7], we can take its delooping, $B^\oplus\mathrm{O}$, the topological category with a single object $*$ and morphism space $B\mathrm{O}^\infty_{\amalg}$, take its homotopy-coherent nerve $\mathrm{N}^{\mathrm{hc}}(B^{\boxplus}\mathrm{O})$, which is a quasi-category, and finally 'loop' again by passing to the under-∞-category under $*$:

$$\mathcal{V}^{\hookrightarrow} := */\mathrm{N}^{\mathrm{hc}}(B^\oplus\mathrm{O}).$$

As we will note, this order of operation is crucial in order to obtain the desired object, in the sense that taking $\mathrm{N}^{\mathrm{hc}}(*/B^\oplus\mathrm{O})$ instead, the nerve of the topological under-category, 'forgets' the topology (see Remark 4.3.2). In $\mathcal{V}^{\hookrightarrow}$, one has the objects of $B\mathrm{O}^\infty_{\amalg}$, and a 1-morphism

$$V \to K$$

from a rank-n vector space V to a rank-$(n+m)$ vector space K is exactly the choice of a rank-m vector space W *and* a path

$$W \oplus V \to K$$

in $B\mathrm{O}(n+m)$.[8] This resembles the idea of $\mathcal{EX}$ we discussed above, and making this resemblence precise will be a major topic of this dissertation. We prove that $\mathcal{V}^{\hookrightarrow}$ adds no more non-invertible paths to $B\mathrm{O}^{\infty}_{\amalg}$:

Theorem (4.3.11). *$\mathcal{V}^{\simeq} \simeq B\mathrm{O}^{\infty}_{\amalg}$.*

Here, $\mathcal{V}^{\simeq}$ is the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$, a.k.a. its *core*. While the statement seems obvious, it is not immediately obvious how to provide a map, in either direction, that realises such an equivalence. We construct an explicit map

$$\Psi\colon B\mathrm{O}^{\infty}_{\amalg} \to \mathcal{V}^{\simeq}$$

that we then prove is an equivalence by indirect means. We develop the technology required to provide an explicit inverse – assuming one does not wish to effectively go through the construction constituting the proof of Whitehead's theorem – only later, in Chapter 5. The definition of Ψ is also a warm-up to the less trivial *depth*-1 version of it, which we will discuss momentarily.

The proof applies mutatis mutandis to the delooping of any topological monoid M whose only invertible point is its unit. That is, the proof shows in this case that there is an equivalence

$$(*/\mathrm{N}^{\mathrm{hc}}(BM))^{\simeq} \simeq M$$

of ∞-groupoids. We interpret this to suggest $*/\mathrm{N}^{\mathrm{hc}}(-)$, with slightly misleading terminology, as a *stratified loop space* functor which creates non-invertible paths in M that depend on the monoidal structure. It is thus an inverse to B only at the level maximal sub-∞-groupoids. It remains desirable to understand how, and whether, the idea generalises to accomodate non-strictified structures, and what an eventual stratified Recognition Principle, and a useful definition of spectrum with quasi-categories appearing in this way, may then look like.

In Section 6.1, we finally arrive at a definition of a quasi-categorical incarnation of $\mathrm{T}^{\mathrm{fib}}$ (in the nonrelative case) that circumvents the theory of conically-smooth stratified spaces. Given a linked *manifold* $\mathfrak{S}$, a certain collection of classifiers organise into a span map (see Construction 6.1.5)

$$\mathrm{T}\mathfrak{S}\colon \mathfrak{S} \to B\mathrm{O}(n,m)$$

from $\mathfrak{S}$ to the *linked Grassmannian*

$$\begin{tikzcd} & B\mathrm{O}(m)\times B\mathrm{O}(n) \arrow[dl, two heads, "\mathrm{pr}"'] \arrow[dr, hook, "\oplus"] & \\ B\mathrm{O}(n) & & B\mathrm{O}(n+m) \end{tikzcd}$$

with the appropriate ranks, which then embeds into $\mathcal{V}^{\hookrightarrow}$ via the *unpacking map*

$$\mathbf{U}\colon \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}.$$

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[8]We are simplifying notation somewhat at the expense of ignoring certain subtleties concerning some non-canonical choices, but, as we will see, these issues turn out to be immaterial (due to the straightforward Lemma 6.1.1).

This map is the sole subject of Chapter 5, with the main result being its *existence*:

THEOREM (5.1.3). *There is a fully-faithful functor* $\mathbf{U}\colon \mathcal{EX}(BO(n,m)) \to \mathcal{V}^{\hookrightarrow}$ *of* ∞*-categories.*

The fact that this map is fully faithful is much easier to see than the fact that it exists. Indeed, Chapter 5 is devoted wholly to its construction, and we note said property only later in the proof of Proposition 6.3.13. Due to the point-set definition of $\mathcal{EX}$, and the cumbersome definition of N^{hc}, the construction is rather lengthy, and utilises some convexity arguments for its key idea. While the construction applies in the generality discussed above with topological monoids, it is not at all clear how to extend it to a purely combinatorial (simplicial) or algebraic context: it seems to depend crucially on translating back and forth between topological spaces and Kan complexes using the classical adjunction between geometric realisation and the singular chains functor. In brief, it remains desirable to obtain a simpler construction of $\mathbf{U}$.

Such questions notwithstanding, the span map $\mathrm{T}\mathfrak{S}$ induces a map

$$\mathrm{T}\mathfrak{S}\colon \mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow}$$

of ∞-categories. We thus transport the tangential theory in the conically-smooth context to the linked context.

Chapter 6 is then devoted to a study of a certain kind of tangential structure in this setting. A given *smooth* (non-stratified) tangential structure

$$F\colon Y \to BO(n)$$

of rank n can be seen as a *stratified* tangential structure

$$F\colon Y \to BO(n) \hookrightarrow \mathcal{V}^{\simeq} \subset \mathcal{V}^{\hookrightarrow}$$

after including $BO(n)$ into $\mathcal{V}^{\hookrightarrow}$. However, stratified spaces with Y-structure[9] are exactly the smooth (trivially-stratified) spaces with Y-structure in the ordinary sense. The simplest and most elegent generalisation of this idea that produces non-trivial results is also due to AFR, who, in [7], consider *solid Y-structures* on stratified spaces, which generalise the idea that if a manifold M is of dimension $n' < n$, then a solid Y-structure ought to be a Y-structure on a rank-n extension of its tangent bundle $\mathrm{T}M$. In the categorical literature, this replacement of $F\colon Y \to \mathcal{V}^{\hookrightarrow}$ is known as a *cartesian fibration replacement*, and we borrow that name for the following definition, which is due to AFR:

DEFINITION (6.3.8). The *cartesian fibration replacement* of $F\colon Y \to \mathcal{V}^{\hookrightarrow}$ is the ∞-functor

$$\overline{F}\colon \overline{Y} = \overline{(Y,F)} = (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y \to (\mathcal{V}^{\hookrightarrow})^{\{0\}},$$

the source evaluation from the fibre product along the target evaluation.

It is a result of [35] that this does indeed give a cartesian fibration replacement of F over $\mathcal{V}^{\hookrightarrow}$.

[9] We mostly suppress F.

The main result of Chapter 6 is a characterisation of cartesian tangential structures on linked manifolds in classical terms:

THEOREM (6.4.20). *A linked manifold possesses a cartesian Y-structure if and only if it possesses a solid Y-structure.*

Here, a *solid Y-structure* means, by our definition unlike AFR's, a solid structure on the individual strata (in the ordinary sense described above), together with a *compatibility (map)* over links: see Definition 6.4.18, which is completed by Remark 6.7.14. It can be expressed as a lift to a particular span of spaces over $B\mathrm{O}(n, m)$ – see Observation 6.4.19.

1.3. Field theory and very generalised homology

We will finally discuss Chapter 7, which has the modest aim of discussing and extending the topology underlying a construction of Lurie from [53, §4.1]. It associates with any input framed disk algebra a functorial field theory, and we allow arbitrary smooth tangential structures as well as defects. Allowing defects turns out to lead away from the conically-smooth theory, and so the full construction awaits the development of a theory of algebras and homology native to the linked setting, not to mention the appropriate target Morita category, all of which we leave to future work. Therefore, the only goal of this chapter is to discuss the underlying topological construction. As will be evident, our approach is heavily influenced by that of [67] and some results from [9, 8].

At chain level, homology theories in a certain generalised sense can be characterised as functors $\mathcal{F}$, on spaces of a certain type, satisfying two properties:

- compatibility with collared cutting and gluing: if $M = M_- \amalg_{M_0 \times \mathbf{R}} M_+$, then $\mathcal{F}(M) \simeq \mathcal{F}(M_-) \otimes^{\mathbb{L}}_{\mathcal{F}(M_0 \times \mathbf{R})} \mathcal{F}(M_+)$, and
- compatibility with exhaustion: if $M = \mathrm{colim}(\emptyset \subset M_0 \subset M_1 \subset \cdots)$, then $\mathcal{F}(M) \simeq \mathrm{colim}_{\mathbf{N}} \mathcal{F}(M_i)$.

For spaces M locally modelled by 'basic disks' with a specified type of stratification and possibly a specified tangential structure (organised in an ∞-category of 'basics' possibly with such structure), Ayala–Francis–Tanaka (AFT) showed in [8] that functors $\mathcal{F}$ (valued in a nice symmetric-monoidal ∞-category $\mathcal{C}$) as above are necessarily given by factorisation homology, $\mathcal{F}(M) = \int_M A_{\mathcal{F}}$, with coefficients in a basic-disks-algebra $A = A_{\mathcal{F}}$ (in $\mathcal{C}$). We will refer to this fact as the *Locality Theorem*. This is about 'homology theory' in that, in essence, A gives rise to a locally-constant factorisation algebra on M, which can be seen as a cosheaf on the Ran space of M, and factorisation homology becomes (0'th) cosheaf homology (i.e., global sections).[10] Moreover, some well-known homology theories, such as singular homology and Goresky–MacPherson intersection homology, can be recovered at least at chain level by factorisation homology.

[10] These ideas go back to the chiral homology of Beilinson–Drinfeld [11]. Their topological incarnation at the level of factorisation *algebras* rather than homology was developed by Costello–Gwilliam in [21, 22] and by Lurie – at both levels – in [50].

The Locality Theorem can be seen as a homological version of a special case of the Cobordism Hypothesis.[11] The *Cobordism Hypothesis* goes back to the work of Baez–Dolan [10],[12] and is sometimes rendered as the statement that a topological quantum field theory, organised as a (higher) functor out of an appropriate fully-extended bordism category into an appropriate higher symmetric monoidal target category $\mathcal{C}$, is determined by its value on a point.[13] The Locality Theorem is easier to prove than the Cobordism Hypothesis, and so, by the law of the conservation of difficulty, if a translation of the kind we mentioned exists, it should be hard to make precise. The meta-goal of a program, of which we think this work ought to be a part, is to give an express duality between homology theory (in the sense above) and (extended) functorial field theory, such that AFT's theorem translates to the Cobordism Hypothesis. We will now briefly sketch some reasons why this might be expected, and then go on to motivate and explain the contributions of the present work. We will drop the adjectives 'topological' and 'quantum,' and will simply speak of *(extended) functorial field theories*, or *(e)FFTs*.

In [53, §4.1], dissatisfied with the Cobordism Hypothesis due to its calculational impractibility, Lurie proposed a way to construct an eFFT that can be described rather concretely: one would take as input an $\mathbb{E}_n$-algebra A ([56, 55, 50]), or, equivalently, an n-dimensional unstratified disk algebra with tangential structure given by framings (see [36] for a quick exposition), and as output would produce a symmetric-monoidal (∞, n)-functor $\mathcal{Z}_A$ on the n-dimensional fully extended bordism category with stable n-framings with values in 'the' Morita category of $\mathbb{E}_n$-algebras (see also [44, 43] for a history and discussion of the scare quotes). The idea of this construction in terms of factorisation algebras as worked out in Scheimbauer's thesis [67] is explained briefly in [70]; another friendly introduction is [1].

It is, in essence, the iterated application of the following basic idea: let M be a manifold with, for simplicity, a single boundary component $\partial = \partial M$, and say, again for simplicity, that the top dimension of the bordism category is n, and M is n-dimensional. A stable n-framing on M induces a framing on M°, and a framing on $\partial \times \mathbf{R}$, the *framed collar* of the boundary. Consequently, $\int_\partial A$, which is by definition $\int_{\partial \times \mathbf{R}} A$, is naturally an $\mathbb{E}_1$-algebra due to the $\mathbf{R}$-factor (and the functoriality of $\int_{\_} A$), whereas $\int_{M^0} A$ is merely an $\mathbb{E}_0$-algebra in $\mathcal{C}$, i.e., a pointed object therein, its pointing coming from the inclusion of the empty subset (and again the functoriality of $\int_{\_} A$). These two algebras that M and A give rise to are related by an action, one of $\int_\partial A$ on $\int_{M^\circ} A$, parametrised by embeddings of the collared boundary into the interior. In other words, they couple to give a factorisation algebra on the half-line $\mathbf{R}_{\geq 0}$

[11]The idea of applying techniques of stratified space theory and factorisation homology to functorial field theory is certainly not new: this was discussed already in [4, §1]; see also [5].
[12]See Freed [34] for a friendly introduction.
[13]This is in reference to an especially simple unstructured case of the statement. A better (and still unstructured) reformulation is that such TQFTs correspond to the fully-dualisable objects of $\mathcal{C}$.

that is locally-constant with respect to its boundary stratification. This can be realised – and here we digress slightly from the standard account – by pushing forward the factorisation algebra induced by A on M° to $\mathbf{R}_{\geq 0}$ by sending all of 'M' to $*$, and by projecting $\partial \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$. More precisely, this is a stratified map only upon refining M° with a new stratification induced by the closed submanifold $\partial \cong \partial_+ \subset M^\circ$ given by ∂ pushed inwards along a nowhere-vanishing inward-pointing normal vector along ∂ for some positive time, which gives

$$M^\circ \cong M \amalg_{\partial_+} \partial \times \mathbf{R}_{\geq 0},$$

which in turn yields the refinement

$$\overline{M} \to M^\circ$$

with domain consisting of the three strata M°, ∂_+, and $\partial \times \mathbf{R}_{>0}$. The ensuing projection

$$p = p_M \colon \overline{M} \to M^! \coloneqq \mathbf{R}_{\geq 0}$$

defines the field theory associated, in this approach, with the input algebra A, by setting

$$\mathcal{Z}_A(M) \coloneqq (p_M)_*(A).$$

The notation obfuscates the dependence on tangential structures, but this is understood. More specifically, the *Pushforward Theorem* of AFT [8, Theorem 2.25] applies to the constructible bundle p_M and defines $\mathcal{Z}_A(M)$ in the setting above.[14]

If there are two (groups of) boundary components – say 'incoming' and 'outgoing' –, we can proceed similarly and push the algebra forward to $\mathbf{R}_{\{0\}}$, to the real line stratified by a distinguished point. In higher codimensions, the collars have higher-dimensional n-framed collars, so one obtains algebras on euclidean spaces with flag-like stratifications: see [67]. On a point $*$, the collar is merely $\mathbf{R}^n = * \times \mathbf{R}^n$, and the pushforward of A along the projection $* \times \mathbf{R}^n \to \mathbf{R}^n$ gives A itself, so that $\mathcal{Z}_A$ illustrates the Cobordism Hypothesis by being determined by its value on $*$.

Moreover, the Locality Theorem and the Cobordism Hypothesis combine to imply that homology theories on framed n-manifolds correspond exactly to eFFTs on the stably-framed bordism category with values in the Morita category of $\mathbb{E}_n$-algebras. More interestingly, the functoriality of the rule $M \mapsto p_M$ (again suppressing the choice of tangential structure) along the 1-extended bordism category translates to the compatibility with collared cutting and gluing, which we invite the reader to check in low dimensions.

This approach is beset with a number of technical difficulties. The definitions of the various extended bordism categories are much less easy to work with than the classical global (1-extended) Atiyah(–Segal–Witten) formalism ([2, 3, 69, 81]), leading to various theorems and theorem sketches but rather few examples. A worked-out definition finally appeared in Calaque–Scheimbauer [15] after [67], and put to use in [14]; variants and extensions appeared in [40,

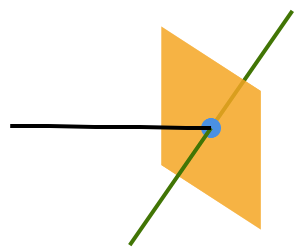
[14]The algebra A is defined on M°, and the projection $M^\circ \to \mathbf{R}_{\geq 0}$ is *weakly* constructible, becoming constructible along the refinement $\overline{M} \to M^\circ$ as discussed.

41]. However, at least as far as FFTs associated with disk algebras are concerned, these aspects seem to distract somewhat from the essential topological operation described above.

Indeed, the way we have presented the construction is non-standard: we are not working with bordisms defined by cut functions on a smooth manifold (without boundary) of full dimension, and we rely on the theory of factorisation homology on stratified spaces to give a one-line construction of $\mathcal{Z}_A$, at least in codimension 1. From this point of view, it is clear that the construction is valid for an input algebra A with *any* tangential structure, provided that M is stably-structured in the same manner. What is essential is the projection p_M associated with it, and that the 'collar' $\overline{M}$, a refinement of the interior[15] possess the tangential structure in question in the usual sense, so that the Pushforward Theorem applies.

Indeed, we systematise the rule $M \mapsto p_M$ in depth 1 in Definition 7.2.2, and call it the P^2 *construction*. It summarises the ideas discussed above and is quite straightforward in its linked formulation. It is preceded by a necessary preliminary section on maps of linked spaces, Section 7.1. Upon linked realisation, the P^2 construction recovers the ordinary stratified version of the story. Once this has been achieved, there are two further directions that remain to be dealt with.

First, defects. Factorisation homology on stratified spaces with tangential structure is perfectly capable of evaluating defect submanifolds, not just boundary components. Therefore, in an approach that uses this theory, the P^2 *construction*, and the ensuing FFT construction $A \mapsto \mathcal{Z}_A = (M \mapsto (p_M)_* A)$ should therefore have an extension that takes into account defect submanifolds within M.[16] We describe such an extension in Section 7.4, and see that the projections can have targets such as



upon realisation – see Example 7.4.8. The stratified spaces thus obtained are conically smooth (see Remark 7.4.9), and AFT's Pushforward Theorem can be applied to the weakly constructible bundles p, defining the TQFT.

We then also discuss cutting and gluing, which is 'dual' to functoriality, and obtain the following

[15] possibly times some euclidean factor

[16] The idea of connecting factorisation algebras/homology, or stratified space theory, with defect coupling is one that is very much in the air at the time of writing. There is some progress in BV(-BFV)-type contexts: see [18] and the references therein.

Proposition (7.5.5). *The P^2 construction is compatible with cutting and gluing.*

In the linked setting where we formulate it, this proposition has a completely straightforward proof. Our treatment of cutting-and-gluing is quite on-the-nose and conforms to that in more physically-minded literature, such as e.g. in [16], much closer to the Atiyah formalism.

Until Section 7.6, P^2 does not refer to any tangential structure, and so works with any at the price of being insensitive to it. In order to formulate a sensitive version, we define in Definition 7.6.1 a *stable* Y-structure on a linked manifold based on our result on cartesian structures on linked manifolds. We then note in Lemma 7.6.3 that the construction can be applied to bordisms with defects equipped with a stable Y-structure: see Definition 7.6.4.

We conclude with Section 7.7, which contains a brief discussion the following problem: It is not clear how to extend the P^2 construction to bordisms with defects that only possess a cartesian structure. We discuss the reason why, and a trivial and unsatisfactory remedy. A version of the construction *does* apply to bordisms with cartesian structure if they do not have defects: see Remark 7.7.2.

1.4. Conventions

We list some of our conventions below. More will be fixed in Chapter 2.

- The set $\mathbf{N}$ of natural numbers includes zero.
- We denote the real line by $\mathbf{R}$.
- We denote by $\boldsymbol{\Delta}$ the simplex category, and its objects by $[n]$, $n \in \mathbf{N}$. The standard n-simplex is the simplicial set $\Delta[n] = \mathrm{Hom}_{\boldsymbol{\Delta}}(-, [n])$, and we employ the Yoneda Lemma without mention.
- 'Coface' and 'codegeneracy' maps we simply call 'face' and 'degeneracy' maps.
- We say ∞-*category* to mean a quasi-category, and ∞-*groupoid* to mean a Kan complex.
- Cartesian products of simplicial sets are defined dimension-wise.
- Given two simplicial sets $\mathcal{C}, \mathcal{D}$, we write $\mathcal{C}^{\mathcal{D}} = \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ for the simplicial set whose set $\left(\mathcal{C}^{\mathcal{D}}\right)_k$ of k-simplices is the set of maps $\mathcal{D} \times \Delta[k] \to \mathcal{C}$ of simplicial sets, together with the obvious simplicial maps.
- A *cofibration* of simplicial sets is a monomorphism.
- For $x \in \mathcal{C}$ an object, we write $\mathcal{C}/x$ rather than $\mathcal{C}_{/x}$ for the over-∞-category over x, and similarly $x/\mathcal{C}$ for the under-∞-category.
- A *map* of ∞-categories is a map of the underlying simplicial sets, i.e., a natural transformation between the two set-valued presheaves on $\boldsymbol{\Delta}$. Occasionally, we call such a map an ∞-*functor*.
- Given maps $f\colon \mathcal{A} \to \mathcal{C}$ and $g\colon \mathcal{B} \to \mathcal{C}$ of ∞-categories, we write $(f \downarrow g) = (f \downarrow g)^{\mathcal{C}}$ or, if the maps are understood, $(\mathcal{A} \downarrow \mathcal{B}) = (\mathcal{A} \downarrow \mathcal{B})^{\mathcal{C}}$ for the ∞-categorical comma category construction (called *oriented*

fibre product in [52]) defined to be the iterated fibre product

$$(f \downarrow g) = \mathcal{A} \times_{\mathcal{C}^{\{0\}}} \mathcal{C}^{\Delta[1]} \times_{\mathcal{C}^{\{1\}}} \mathcal{B}.$$

- An *equivalence* of ∞-categories is an equivalence-of-∞-categories as in [52].
- When a topological space X appears in place of an ∞-category, we mean the ∞-groupoid $\mathrm{Sing}_\bullet(X)$ of its singular chains.
- By a *Kan-enriched category* we mean a locally Kan category, i.e., a simplicially-enriched category whose morphism spaces are Kan complexes.
- *Smooth* manifolds have no boundary.
- We denote the trivial real rank-k vector bundle over a given space by ε^k.

CHAPTER 2

Preliminaries

2.1. Quasi-categories

Quasi-categories are to stratified spaces as Kan complexes are to spaces.

2.1.1. Kan complexes and spaces. To expand briefly,[1] let

$$\Delta^n \coloneqq \Big\{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} : \sum x_i = 1\Big\}$$

be the *standard topological n-simplex*, the convex hull of the standard basis of $\mathbf{R}^{n+1}$. For X a topological space, let $X_n \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$ be the set of continuous maps from Δ^n to X, called the set of *n-simplices* of X. There are natural maps between the sets in this collection induced by pulling back along (pre-composing with) maps of type $\Delta^n \to \Delta^m$ for varying n and m. Maps of the latter kind can themselves be given in terms of which corner, i.e., basis vector, in Δ^n is mapped to which corner in Δ^m, and so in terms of functions of type $\{0, \dots, n\} \to \{0, \dots, m\}$. Asking these latter maps to respect the natural orientations of the standard topological simplices amounts to asking that they be non-decreasing, which leads us to the *simplicial set* structure on the collection $(X_n)_{n \in \mathbf{N}}$. That is, abstracting away from the space, we have arrived at (most of) the following

Definition. A *simplicial set* is a collection $S = S_\bullet = (S_n)_{n\in\mathbf{N}}$ of sets together with a map $\phi^*\colon S_m \to S_n$ for every order-preserving map $\phi\colon \{0 \le \cdots \le n\} \to \{0 \le \cdots \le m\}$, such that $\phi^*\psi^* = (\psi \circ \phi)^*$ whenever ψ and ϕ compose.

In other words, a simplicial set is a functor $S\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, where $\mathbf{\Delta}$ is the *simplex category*, whose objects are the finite ordinals $[n] \coloneqq \{0 \le \cdots \le n\}$, $n \in \mathbf{N}$, and whose morphisms are given by order-preserving maps. The target Set is the category of sets and functions. One writes $S_n \coloneqq S([n])$. The maps ϕ^* are called *simplicial maps*. A *map* of simplicial sets is a natural transformation. The resulting category of simplicial sets is denoted by $\mathrm{sSet} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$.

2.1.2. Geometric realisation. The collection (X_n) induced by a topological space X as discussed above can be expressed as the simplicial set

$$\mathrm{Sing}_\bullet(X)\colon \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{|-|} \mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Top}}(-,X)} \mathrm{Set}$$

[1] We assume that the reader is familiar with ordinary category theory, including the Yoneda Lemma, adjunctions, Kan extensions, and enrichment (see, e.g. [54, 64]), as well as with basic homotopy theory.

and is called the *complex of singular chains* of X. Here, $|-|$ is called the *geometric realisation* functor, defined by $[n] \mapsto |[n]| \coloneqq \Delta^n$ and

$$(\phi\colon [n] \to [m]) \mapsto \left((t_0, \dots, t_n) \mapsto \left(\sum_{\phi(t_i)=0} t_i, \dots, \sum_{\phi(t_i)=m} t_i \right) \right),$$

where empty sums are understood to be zero. Less trivially, and more commonly, $|-|$ has a left Kan extension

$$|-|\colon \mathrm{sSet} \to \mathrm{Top}$$

along the Yoneda embedding $\Delta(-)\colon \mathbf{\Delta} \to \mathrm{sSet}$. The latter is given by

$$[n] \mapsto (\Delta[n] \coloneqq \Delta([n])\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{sSet},\ [n] \mapsto \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]))\,.$$

This being an extension, we have $|\Delta[n]| = |[n]| = \Delta^n$. It exists because Top is cocomplete and so can be computed as follows:

$$|S| = \mathrm{colim}_{\Delta[n] \to S} \Delta^n.$$

This is a non-sensical expression that abbreviates the colimit in Top of the diagram

$$(\mathbf{\Delta} \downarrow S) \to \mathrm{Top}$$

where $(\mathbf{\Delta} \downarrow S)$ is the comma category taken within sSet of the functors $\Delta(-)$ and $\{S\} \hookrightarrow \mathrm{sSet}$, that is, the category whose objects are maps $\Delta[n] \to S$ of simplicial sets and whose morphisms from $\Delta[n] \to S$ to $\Delta[m] \to S$ are maps $\Delta[n] \to \Delta[m]$ such that the resulting triangle commutes. The functor $(\mathbf{\Delta} \downarrow S) \to \mathrm{Top}$ is the composition $(\mathbf{\Delta} \downarrow S) \to \mathbf{\Delta} \to \mathrm{Top}$ given by first forgetting to domains and then applying the original $|-|$.

We have thus set up the functors $\mathrm{Sing}_\bullet(-)\colon \mathrm{Top} \rightleftarrows \mathrm{sSet}\colon |-|$. In fact, $|-|$ is left adjoint to $\mathrm{Sing}_\bullet(-)$, as can be seen by applying the formula for $|-|$. When S is the complex of singular chains of a topological space, then the adjunction yields natural bijections $\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], S) \cong S_n$. More generally, this is implied for any simplicial set S by the Yoneda Lemma. Much more can be said about this adjunction, but need not be. See [39] for a textbook account.

2.1.3. Composition and contractibility. Let us observe now that a simplicial set S is a relaxed kind of (small) category: the set S_0 is called the set of its *vertices* or *objects*, and the set S_1 is called the set of its *edges* or *morphisms*. The *source and target maps* are the pullbacks along the maps $[0] \to [1]$ given by $0 \mapsto 0$ and $0 \mapsto 1$, respectively. The *identity morphism* $\mathrm{id}_x \in S_1$ at $x \in S_0$ is given by pulling x back along $[1] \to [0]$, $0, 1 \mapsto 0$. However, the straightforward renamings end here since S carries no map that emulates composition of morphisms. Instead, one says that *a* composition of $f\colon x \to y$ and $g\colon y \to z$ is *witnessed* by a 2-simplex $H \in S_2$ whose pullback along $[1] \overset{\mathrm{id}}{\hookrightarrow} [2]$ and $[1] \xhookrightarrow{\mathrm{id}+1} [2]$ is f and g, respectively. In this case, H is said to witness its pullback along $[1] \to [2]$, $0 \mapsto 0$, $1 \mapsto 2$ as *a composition* of f and g.

The simplicial set S will emulate category-like composability of morphisms if, given f and g as above, an H as above exists uniquely. However, for the *associativity* of composition, and its 'coherence,' we will have to consider witnesses of higher and higher dimensions. If all such witnesses exist uniquely, then S will uniquely determine, and be determined by, a category. But since our aim is not just to re-express category theory, let us consider the question as to how we can organise composition when H does exist but *not* uniquely.

Given f and g, there is a whole simplicial set of witnesses H. In order to express it, note first that the simplicial set of composable morphisms in S can be written as the pullback

$$\begin{tikzcd} \mathrm{Comp}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\Delta[1], S) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Fun}(\Delta[1], S) \arrow[r, "\mathrm{ev}_0"] & \mathrm{Fun}(\Delta[0], S) \end{tikzcd}$$

where every term is a simplicial set. Indeed, in contrast to $\mathrm{Hom}_{\mathrm{sSet}}(-,-)$, which gives sets, one writes $\mathrm{Fun}(A,B) = \mathrm{Fun}_\bullet(A,B)$ for the simplicial set whose set of n-simplices is $\mathrm{Hom}_{\mathrm{sSet}}(A \times \Delta[n], B)$, and whose simplicial maps are induced by applying them on the $\Delta[-]$-factor. The maps ev_1 and ev_1 are induced by taking source and target, respectively. Now, observe that a 2-simplex of S provides, as discussed above, a pair of morphisms one of whose target vertex is the source vertex of the other. Going through the Yoneda Lemma, we obtain the function $\mathrm{Hom}(\Delta[2], S) \to \mathrm{Hom}(\Delta[1]) \times \mathrm{Hom}(\Delta[1], S)$, which trivially extends over the $\Delta[-]$-factor to define a map $\mathrm{Fun}(\Delta[2], S) \to \mathrm{Comp}(S)$. The *simplicial set of compositions* of f and g is then the fibre of this map at $(f,g) \in \mathrm{Comp}(S)_0$, that is, the pullback

$$\begin{tikzcd} \mathrm{Comp}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(\Delta[2], S) \arrow[d] \\ \{(f,g)\} \arrow[r, hook] & \mathrm{Comp}(S) \end{tikzcd}$$

where $\{(f,g)\}$ is the simplicial set given by a singleton at each degree, and the map $\{(f,g)\} \to \mathrm{Comp}(S)$ sends the unique vertex to (f,g), and the unique n-simplex to the pullback of (f,g) along the unique map $[n] \to [0]$.

Asking that f and g compose *essentially uniquely*, that is, that the choice of witness be topologically irrelevant, can be made precise by asking that $\mathrm{Comp}(f,g)$ be *contractible*. The notion can be borrowed directly from (its weak version in) classical homotopy theory by expressing spheres in simplicial terms: for each $n \in \mathbf{N}$, there is a simplicial subset $\partial\Delta[n] \subset \Delta[n]$, the *boundary* of $\Delta[n]$, or the *simplicial n-sphere*. It is empty if $n = 0$, and if $n \geq 1$, then it is the simplicial subset generated by the pullbacks $d_i(\mathrm{id}_{[n]})$ of $\mathrm{id}_{[n]} \in \Delta[n]_n$ along the maps $\partial_i \colon [n-1] \hookrightarrow [n]$ that skip $i \in [n]$:

$$\partial_i(x) = \begin{cases} x, & x \leq i-1, \\ x+1, & x \geq i. \end{cases}$$

These are called the *face* maps, pullbacks along which are denoted, as already indicated, by d_i.[2] The *contractibility* of a simplicial set X can thus be expressed by the condition that all simplicial n-spheres in X have fillers, that is, every lifting problem of type

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

admits a solution.

If $\mathrm{Comp}(f, g)$ is contractible for every composable pair (f, g) of morphisms in S, then S is called a *quasi-category*, and we will call it an ∞*-category* in the rest of this work.[3] Contractibility entails non-emptyness, whence composing is always possible.

This conceptual definition is equivalent, by a result of Joyal (proved in [52, 0079]), to the following traditional definition. Let, first, $\Lambda^n_i \subset \Delta[n]$ be the simplicial subset generated by all the faces $d_j \mathrm{id}_{[n]}$ of $\Delta[n]$ except for the i'th one, called the *i'th horn* of $\Delta[n]$.[4] Then S is an ∞-category if and only if every lifting problem of type

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

admits a solution *whenever* $0 < i < n$. For such natural numbers i and n, Λ^n_i is called an *inner horn*, and when $i = 0$ or $i = n$, it is called an *outer horn*. This condition is called the *weak Kan condition*[5] and the analogous condition for all $0 \leq i \leq n$ is called the *Kan condition*. Simplicial sets that satisfy the latter are called *Kan complexes*, and we will also call them ∞*-groupoids*.

Composable pairs of morphisms in S correspond to maps of type $\Lambda^2_1 \to S$. Let now $f\colon x \to y$ be a morphism in S, and consider moreover the identity id_x. These arrange into an outer horn $\Lambda^2_0 \to S$, and if it has a filler $F\colon \Delta[2] \to S$, then F witnesses id_x as a composition of the morphisms f and $d_0 F$, i.e., the latter as a *left-inverse* of f. Similarly, f and id_y give an outer horn $\Lambda^2_2 \to S$, a filler F of which witnesses $d_2 F$ as a *right-inverse* of f. Similar considerations

[2] An equivalent definition (that of [52, 000R]) of $\partial\Delta[n]$ can be given by setting $(\partial\Delta[n])([m]) = \{\alpha \in \mathrm{Hom}_{\boldsymbol{\Delta}}([m], [n]) : \alpha \text{ is not surjective}\}$ and using the obvious simplicial maps.

[3] There are several definitions of what, following Lurie, we call an '∞-category,' and all fall under the less opinionated umbrella term '$(\infty, 1)$-category,' all connected by chains of Quillen equivalences with respect to certain model structures. The term 'quasi-category' is universally accepted to mean what we mean by it. The notion goes back to [13, 78], and was developed alongside another model, that of simplicially enriched categories, which we will discuss below. There are systematic treatments and comparisons of the competing (complementary) definitions. See e.g. [75, 48, 12]. Several other foundational works on the subject will be mentioned later.

[4] Equivalently ([52, 000U]), $\Lambda^n_i([m]) = \{\alpha \in \mathrm{Hom}_{\boldsymbol{\Delta}}([m], [n]) : [n] \not\subseteq \alpha([m]) \cup \{i\}\}$.

[5] Now antiquated, ∞-categories were originally called *weak Kan complexes*.

in all dimensions justify the term '∞-groupoid' for Kan complexes. If $S = \mathrm{Sing}_\bullet(X)$ for a topological space X, then S is a Kan complex, since such inversion of paths up to homotopies which are witnessed by higher paths is possible within X.

2.1.4. Quasi-categories and stratified spaces. A *stratified space*, in its purest form,[6] is a topological space X together with a continuous map $s\colon X \to \mathcal{P}$, called its *stratification*, to a poset $\mathcal{P}$, called its *stratifying poset*, equipped with the Aleksandrov topology, in which downward-closed subsets are declared closed. The subsets $X_p := s^{-1}p \subseteq X$ are called its *strata*,[7] and X_p is called *lower* than X_q, and the latter *higher* than the former, if $p \leq q$. The length of the longest non-trivial sequence of arrows in $\mathcal{P}$ is called its *depth*.

The standard topological simplex Δ^n has a natural stratification over $[n]$, given by writing it as the n-fold closed cone on a singleton stratified over the trivial poset. At every iteration of this closed-cone taking, a minimal object is adjoined to the stratifying poset. We recall this in more detail in Remark 3.1.10, and will concentrate in the present section on $\Delta^1 \cong [0,1]$ alone. In this scheme, its stratification $s\colon \Delta^1 \to [1]$ is given by $s(0) = 0$ and $s(t) = 1$ for $t > 0$.

A *map* from $s_X\colon X \to \mathcal{P}$ to $s_Y\colon Y \to \mathcal{Q}$ is a poset map $s_f\colon \mathcal{P} \to \mathcal{Q}$ and a continuous map $f\colon X \to Y$ covering s_f. Consequently, a *stratified path* in $X \to \mathcal{P}$, a map (γ, s_γ) from $\Delta^1 \to [1]$ to $X \to \mathcal{P}$, is an ordinary path γ with image within at most two strata, $s_\gamma(0)$ and $s_\gamma(1)$, such that

$$s_\gamma(0) = s_X(\gamma(0)) \leq s_X(\gamma(t)) = s_X(\gamma(t')) = s_\gamma(1)$$

whenever $0 < t, t' \leq 1$. Consequently, if $s_\gamma(0) \prec s_\gamma(1)$ strictly in $\mathcal{P}$, then there is *no* inverse stratified path starting at $\gamma(1)$ and ending at $\gamma(0)$. The corresponding version of $\mathrm{Sing}_\bullet$ that takes stratifications into account and therefore contains non-invertible paths will still be a simplicial set, but not a Kan complex, rather only an ∞-category.

Example 2.1.1. The stratified 'identity map' of $\Delta^1 \to [1]$, the pair $(\mathrm{id}_{\Delta^1}, \mathrm{id}_{[1]})$, is valid but not invertible.

Example 2.1.2. Consider $s\colon \mathbf{R} \to \{0 \prec -, +\}$ given by $s(0) = 0$, $s(\mathbf{R}_{<0}) = \{-\}$ and $s(\mathbf{R}_{>0}) = \{+\}$. This is an example of a depth-1 stratified space. The preimages of downward-closed subsets are $\{0\}$, $\mathbf{R}_{\leq 0}$, and $\mathbf{R}_{\geq 0}$, which are all closed, so we have a stratified space. Stratified paths can travel from 0 to $\mathbf{R}_{<0}$ or $\mathbf{R}_{>0}$, but there are no stratified paths between $\mathbf{R}_{<0}$ and $\mathbf{R}_{>0}$. In particular, being-in-the-same-stratified-path-connected-component is not a transitive relation.

[6] We concentrate in this section on what are known as *poset-stratified* spaces. They are, at least in this work, to the concept of stratified space as quasi-categories are to the concept of $(\infty, 1)$-category.

[7] These need not be connected – we do not require strata to be connected in this work. Sometimes in the literature, the connected components of the X_p are referred to as strata.

Example 2.1.3. Consider $s\colon \mathbf{R}^3 \to [2]$ given by $s(0) = 0$, $s(V \smallsetminus \{0\}) = \{1\}$ where V is a line through the origin, and $s(\mathbf{R}^3 \smallsetminus V) = 3$. The preimage of a downward-closed subset is either V or $\mathbf{R}^3$ and so is closed. This is an example of a depth-2 stratified space.

If one considers a version of $\mathrm{Sing}_\bullet(X)$ by asking the simplices to respect the stratification, then one obtains the simplicial set $\mathrm{Exit}_\bullet(X)$, which is an ∞-category if X is well-behaved – see Section 1.1 for a brief review and the references. To conclude this section, we will recall a more combinatorial characterisation of the functoriality involved in the definition of a simplicial set, which we will use throughout the text for practical purposes, and on which our own definition of the exit path ∞-category will be explicitly based.

Let $S = S_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. There is a special class of morphisms in Δ, the functoriality of S along which is equivalent to its global functoriality. They consist of the *face maps* (which we already mentioned above) and the *degeneracy maps*. Collectively, we call them the *simplicial operators*.

The face maps are of type

$$\partial_i = \partial_i^n\colon [n-1] \hookrightarrow [n]$$

for $i \in [n]$, which is defined to be the unique monotone surjection onto $[n] \smallsetminus \{i\}$. The degeneracy maps are of type

$$\sigma_i = \sigma^n\colon [n+1] \to [n]$$

for $i \in [n]$, which is defined to be the unique monotone surjection onto all of $[n]$ such that $\sigma_i(i) = \sigma_i(i+1) = i$.

Along S, these maps introduce the maps

$$d_i\colon S_n \to S_{n-1}$$

and

$$s_i\colon S_n \to S_{n+1}$$

respectively, which we also call the *i'th face map* and the *i'th degeneracy map*.

The datum of S is equivalent the collection $(S_n, d_i, s_i)_{n,i}$ such that the *simplicial identities* are satisfied, which are listed in the proof of Lemma 3.2.3. Moreover, a map of simplicial sets is equivalent to a collection degree-wise set maps that commute with the face and degeneracy maps.

2.2. Basic constructions

Let $\mathcal{C}\mathrm{at}_{\boldsymbol{\Delta}}$ denote the category of simplicial categories, that is, the category of categories enriched in sSet. We assume the reader is familiar with the nerve $\mathrm{N}(C) = \mathrm{N}_\bullet(C) \in \mathrm{sSet}$ of an ordinary category C.

2.2.1. The homotopy-coherent nerve. We will first recall the simplicial nerve construction ([19], though see also [52, 00KT]), following [51, §1.1.5]. We will then recall its mirror image, the homotopy-coherent nerve, that will feature heavily in Chapter 4. Our constructions and results hold, mutatis mutandis, equally well for either choice.[8]

Similarly to the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{sSet}$, $[k] \mapsto \Delta[k]$, which gives a simplicial set for each $k \in \mathbf{N}$, there exists a functor

$$\mathfrak{C}\colon \mathbf{\Delta} \to \mathcal{C}\mathrm{at}_{\mathbf{\Delta}}.$$

Definition 2.2.1. We first define $\mathfrak{C}$ on objects, then on morphisms.

(1) The simplicial category $\mathfrak{C}[k]$ has the same objects as those of $[k]$, and the simplicial sets of morphisms in each $\mathfrak{C}[k]$ are given by

$$\mathrm{Hom}_{\mathfrak{C}[k]}(i,j) \coloneqq \mathrm{N}(P_{i,j}),$$

where $P_{i,j}$, $0 \leq i, j \leq k$ is empty if $i > j$, and

$$P_{i,j} = \{I \subseteq \{i \leq a+1 \leq \cdots \leq j\} \subseteq [k] : a, b \in I\}$$

if $i \leq j$. In other words, $P_{i,j}$ is the poset consisting of the subposets of $[k]$ that start at i and j, with partial order $\preceq$ given by subset inclusions.

For each triple $i \leq j \leq p$ in $[k]$, there is a map

$$P_{j,p} \times P_{i,j} \to P_{i,p}$$

defined by taking unions. The ordinary nerve functor applied to these maps yields maps

$$\mathrm{Hom}_{\mathfrak{C}[k]}(j,p) \times \mathrm{Hom}_{\mathfrak{C}[k]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[k]}(i,p)$$

of simplicial sets, which is associative since so is taking unions.

(2) A map $f\colon [l] \to [k]$ in $\mathbf{\Delta}$ induces a map $\mathfrak{C}[l] \to \mathfrak{C}[k]$ as follows: on objects, it is given by $[l] \ni i \mapsto f(i) \in [k]$, and on the mapping posets it is given by $P_{i,j} \ni I \mapsto f(I) \in P_{f(i),f(j)}$, applying N to which defines the map $f = \mathfrak{C}f\colon \mathfrak{C}[l] \to \mathfrak{C}[k]$.

Definition 2.2.2. We call the $P_{i,j}$ *mapping posets*, and their nerves *mapping spaces*.

Definition 2.2.3. The *simplicial nerve* $\mathrm{N}^{\mathbf{\Delta}}(\mathcal{D}) = \mathrm{N}^{\mathbf{\Delta}}_{\bullet}(\mathcal{D})$ of a simplicial category $\mathcal{D}$ is the simplicial set whose set of k-simplices is defined by

$$\mathrm{N}^{\mathbf{\Delta}}_{k}(\mathcal{D}) \coloneqq \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{\Delta}}}(\mathfrak{C}[k], \mathcal{D}).$$

This is contravariant in $[k]$ via the covariance of $\mathfrak{C}$.

In other words, $\mathrm{N}^{\mathbf{\Delta}}$ is the restriction of the Yoneda embedding $\mathcal{C}\mathrm{at}_{\mathbf{\Delta}} \to \mathrm{pSh}(\mathcal{C}\mathrm{at}_{\mathbf{\Delta}})$ along $\mathfrak{C}\colon \mathbf{\Delta} \to \mathcal{C}\mathrm{at}_{\mathbf{\Delta}}$. We also write $\mathrm{Fun}(-,-)$ to take the set of functors between the arguments.

Definition 2.2.4. The *homotopy-coherent nerve* $\mathrm{N}^{\mathrm{hc}}(\mathcal{D}) = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{D})$ of a simplicial category $\mathcal{D}$ is the simplicial set whose set of k-simplices is defined by

$$\mathrm{N}^{\mathrm{hc}}(\mathcal{D})_k \coloneqq \mathrm{Fun}(\mathrm{Path}[k], \mathcal{A}),$$

where $\mathrm{Path}[k] \coloneqq \mathfrak{C}[k]^{\mathrm{op}}$. We write $\geq$ for the partial order thereon.

[8]However, see [46] for a cautionary tale.

Recall that, for any category C, we have an isomorphism $\mathrm{N}_\bullet(C^{\mathrm{op}}) \cong (\mathrm{N}_\bullet C)^{\mathrm{op}}$ ([52, 003Q]), where the RHS is the ∞-categorical opposite. Thus,

$$\mathrm{Hom}_{\mathrm{Path}[k]}(i,j) = \mathrm{N}(P^{\mathrm{op}}_{i,j}),$$

and we keep this superscript 'op' throughout. This redundancy is meant to provide clarity. In [51], Lurie writes P for our P, while in [52] he writes P for our P^{op}. As an admittedly sub-optimal middle ground, we exclusively work with N^{hc} (as in [52]), but we keep the superscript.

Functors of type $\mathrm{Path}[k] \to \mathcal{A}$ are called *k-paths* in $\mathcal{A}$.

If $\mathcal{D}$ is Kan-enriched, then $\mathrm{N}^{\mathbf{\Delta}}(\mathcal{D})$ is an ∞-category by [51, Proposition 1.1.5.10]. The same holds for $\mathrm{N}^{\mathrm{hc}}(\mathcal{D})$ by [52, 00LJ]. Both are variants of a result of Cordier–Porter [20].

2.2.2. Joins and (co)slices. For $f\colon K \to \mathcal{C}$ a functor from a simplicial set to an ∞-category, there is ([47], [52, 01GP]) a right fibration $\mathcal{C}/f \to \mathcal{C}$ and a left fibration $f/\mathcal{C} \to \mathcal{C}$, whose domains are respectively called the *slice* and *coslice* of $\mathcal{C}$ at f. We will recall their definitions, but refer the reader to the op. cit. for the named lifting properties.

In the following, our convention is that $X_{-1} = \emptyset$ for $X_\bullet$ a simplicial set, and a product is empty if one of its factors is $\emptyset$. The following is equivalent to the more standard definition; see [52, 0234]. It is a simplicial version of Milnor's general topological construction from [58].

Definition 2.2.5. The *join* $X \star Y = (X \star Y)_\bullet$ of two simplicial sets $X = X_\bullet$, $Y = Y_\bullet$ is defined by

$$(X \star Y)_k = \{(\pi, f_-, f_+) : \pi\colon \Delta[k] \to \Delta[1], f_-\colon \Delta[k]|_0 \to X, f_+\colon \Delta[k]|_1 \to Y\},$$

where π, f_-, f_+ are maps of simplicial sets, and $\Delta[k]|_i = \{i\} \times_{\Delta[1]} \Delta[k]$, $i = 0, 1$, is defined using π. Given $\phi\colon [l] \to [k]$ in $\mathbf{\Delta}$, the corresponding $\phi\colon \Delta[l] \to \Delta[k]$ defines a map $(X \star Y)_k \to (X \star Y)_l$ by restrictions.

Remark 2.2.6. We have injections

$$\iota_0\colon X \hookrightarrow X \star Y, \quad \iota_1\colon Y \hookrightarrow X \star Y.$$

For the former, let $f\colon \Delta[k] \to X$ be a k-simplex of X. Defining

$$\pi\colon \Delta[k] \to \{0\} \hookrightarrow \Delta[1]$$

and setting $f_- = f$, and necessarily $f_+\colon \emptyset \to Y$, gives a map $X_k \to (X \star Y)_k$. In the inclusion of Y into $X \star Y$, π is defined by factoring through the projection to 1 and setting f_- empty instead.

Remark 2.2.7. The join construction is functorial in both arguments. Given $\phi\colon X \to X'$, $\psi\colon Y \to Y'$, we write $\phi \star \psi$ for the induced map $X \star Y \to X' \star Y'$.

Definition 2.2.8. Let K be a simplicial set, $\mathcal{C}$ an ∞-category, and $f\colon K \to \mathcal{C}$ a map. The *slice* $\mathcal{C}/f$ of $\mathcal{C}$ at f is the simplicial set defined by

$$(\mathcal{C}/f)_n = (\mathrm{Hom}_{\mathrm{sSet}})_K(\Delta[n] \star K, \mathcal{C}),$$

where the subscript K indicates that the set in question consists of maps

$$\phi\colon \Delta[n] \star K \to \mathcal{C}$$

whose precomposition

$$K \overset{\iota_1}{\hookrightarrow} \Delta[n] \star K \xrightarrow{\phi} \mathcal{C}$$

is f.

The face and degeneracy maps are given by precomposition and functoriality: a map $\psi\colon \Delta[m] \to \Delta[n]$ induces a map

$$\Delta[m] \star K \xrightarrow{\psi \star \mathrm{id}} \Delta[n] \star K \xrightarrow{\phi} \mathcal{C},$$

which is clearly in $(\mathcal{C}/f)_m$, i.e., $(\phi \circ (\psi \star \mathrm{id}))|_K = f$. The slice is again an ∞-category.

The projection $\mathcal{C}/f \to \mathcal{C}$ is given by precomposing $\phi\colon \Delta[n] \star K \to \mathcal{C}$ with $\Delta[n] \overset{\iota_0}{\hookrightarrow} \Delta[n] \star K$.

The *coslice* $f/\mathcal{C}$ is defined analogously, with $\Delta[n] \star K$ replaced by $K \star \Delta[n]$, ι_1 by ι_0 and vice versa, throughout. It is again an ∞-category.

Notation 2.2.9. Let $\iota_x\colon \Delta[0] \to \mathcal{C}$ be given by a vertex $x \in \mathcal{C}_0$. We write

$$\mathcal{C}/x \coloneqq \mathcal{C}/\iota_x, \quad x/\mathcal{C} \coloneqq \iota_x/\mathcal{C}.$$

They are respectively called the *over-* and *under-∞-category* at x.

Remark 2.2.10. There are canonical isomorphisms

$$\Delta[k] \star \Delta[l] \simeq \Delta[k+1+l],$$

such that the composition

$$\Delta[k] \overset{\iota_0}{\hookrightarrow} \Delta[k] \star \Delta[l] \xrightarrow{\sim} \Delta[k+1+l]$$

is given by

$$[k] \hookrightarrow [k+1+l],\ i \mapsto i,$$

and such that the composition

$$\Delta[l] \overset{\iota_1}{\hookrightarrow} \Delta[k] \star \Delta[l] \xrightarrow{\sim} \Delta[k+1+l]$$

is given by

$$[l] \hookrightarrow [k+1+l],\ i \mapsto k+1+i.$$

Remark 2.2.11. We should explicate the degeneracies in an under-∞-category $x/\mathcal{C}$. Via Remark 2.2.10, a 0-simplex of $x/\mathcal{C}$ is a 1-simplex of $\mathcal{C}$ with source x. Given a 1-simplex

$$\gamma\colon \Delta[0] \star \Delta[1] \to \mathcal{C}$$

of $x/\mathcal{C}$ its source and target γ_0, γ_1, are given, according to Definition 2.2.8, by

$$\gamma_0\colon \Delta[0] \star \Delta[0] \overset{\mathrm{id}\star 0}{\hookrightarrow} \Delta[0] \star \Delta[1] \xrightarrow{\gamma} \mathcal{C},$$

and similarly with $\mathrm{id} \star 1$ for γ_1. The faces (and degeneracies) of simplices of all dimensions can be understood analogously: see Lemma 4.3.4.

CHAPTER 3

Linked spaces and exit paths

Let $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{N}$ be ∞-groupoids. We wish to construct an ∞-category that interprets $\mathcal{L}$ as the space of *non-invertible* paths from $\mathcal{M}$ to $\mathcal{N}$, without modifying the paths of $\mathcal{M}$ and $\mathcal{N}$, and such that vertices remain exactly those of $\mathcal{M} \amalg \mathcal{N}$. To this end, we first need maps $\mathcal{L} \to \mathcal{M}, \mathcal{N}$, which play the respective roles of source and target. For the sake of clarity, we separated the construction into two steps: first we will discuss the 'space' of non-invertible paths, and then adjoin it in a certain way to $\mathcal{M} \amalg \mathcal{N}$.

3.1. Exit shuffles

Definition 3.1.1. Let $\iota\colon \mathcal{L} \to \mathcal{N}$ be a map of simplicial sets. We call the simplicial set $\mathcal{P} \coloneqq \mathcal{P}_\iota \coloneqq \mathcal{L} \times_{\mathcal{N}^{\{0\}}} \mathcal{N}^{\Delta[1]}$ the *mapping cocylinder* of ι.

Remark. Definition 3.1.1 is a variation on the under-∞-category construction, and reduces to it if $\mathcal{L} = \mathrm{pt}$ is the constant singleton, in that there is an equivalence $\iota(\mathrm{pt})/\mathcal{N} \simeq \mathrm{pt} \times_{\mathcal{N}^{\{0\}}} \mathcal{N}^{\Delta[1]}$. Note that otherwise the coslice $\iota/\mathcal{N}$ does not model a space of paths starting in $\mathcal{L}$: its simplices, as simplices of $\mathcal{N}$, are higher-dimensional than required to begin with. Rather, it is the space of cocones under ι.

Remark 3.1.2. Recall how the mapping cocylinder appears in classical topology: in the analogous construction with spaces L, N and ι a continuous map, the natural map $P_\iota \to N$ is a fibration replacement for ι in view of a homotopy equivalence $L \simeq P_\iota$.

Remark 3.1.3. There are two induced maps $\pi, \tau\colon \mathcal{P} \to \mathcal{M}, \mathcal{N}$ defined as the compositions in the diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d, "s"] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[rr, bend left, dashed, "\tau"] \arrow[dd, bend right, dashed, "\pi"'] & \mathcal{N}^{\Delta[1]} \arrow[r] \arrow[d] & \mathcal{N}^{\{1\}} \\
\mathcal{L} \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathcal{N}^{\{0\}} & \\
\mathcal{M} & &
\end{tikzcd}$$

where the map $\mathcal{N}^{\Delta[1]} \to \mathcal{N}^{\{1\}}$ is given by precomposition with $\{1\} \times \Delta[k] \hookrightarrow \Delta[1] \times \Delta[k]$.

Ideally, one would adjoin $\mathcal{P}$, using $\pi\colon \mathcal{L} \to \mathcal{M}$, to $\mathcal{M} \amalg \mathcal{N}$ as the space of non-invertible paths from $\mathcal{M}$ to $\mathcal{N}$, by employing π, τ of Remark 3.1.3 as source and target maps, respectively, but $\mathcal{P}$ does not lend itself to this directly. Instead, we will extract data out of it that does. First, let us delineate the problem in order to motivate the construction to follow.

Remark 3.1.4. A vertex of $\mathcal{P}$ is a path of $\mathcal{N}$ that starts at a point in $\iota(\mathcal{L})$. One may coherently view this as a path which starts in $\mathcal{M}$, by projecting its source down to $\mathcal{M}$ via σ_0, and which, analogously, ends in $\mathcal{N}$ via τ. For higher morphisms, however, a direct generalisation requires unnatural choices: for instance, a 1-morphism in $\mathcal{P}$ may be depicted as

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur, dashed] & \bullet \arrow[u] \end{tikzcd} \tag{3.1.5}$$

where the bottom edge is in $\iota(\mathcal{L})$, and the top edge is in $\mathcal{N}$. (We depict the $\Delta[1]$-coordinate in a k-morphism of $\mathcal{N}^{\Delta[1]}$, i.e., in a map $\Delta[1] \times \Delta[k] \to \mathcal{N}$ of simplicial sets, as the upwards vertical coordinate.) Two of the (non-degenerate) 2-simplices of $\mathcal{N}$ we may extract are

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \arrow[u] \end{tikzcd} \tag{3.1.6}$$

and

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[ur] & \end{tikzcd} \tag{3.1.7}$$

corresponding to the two $(1,1)$-shuffles

$$\Delta[2] \hookrightarrow \Delta[1] \times \Delta[1]$$

à la Eilenberg–Mac Lane–Zilber [31, 30] (see also [52, 00RF]).[1] If we were to add (3.1.5) as a 2-morphism to $\mathcal{M} \amalg \mathcal{N}$, say with source edge the bottom one, then we would have to choose the hypotenuse of the triangle (3.1.6) as the target edge, and the vertical edge as the intermediate $\overline{12}$-edge. But we may equally well make the analogous choice with triangle (3.1.7), declaring the left vertical edge the source. The problem is that *both* types of triangles are required for composition: if we wish later to concatenate, say, a path in $\mathcal{M}$ with a (non-invertible) 1-morphism in $\mathcal{P}$, then we need (assuming there is a lift to $\mathcal{L}$) a triangle of the first type. Similarly, if we wish to concatenate a non-invertible 1-morphism with a path in $\mathcal{N}$, we need a triangle of the second type.

[1]Triangle (3.1.6) is given by the 2-simplex of $\Delta[1] \times \Delta[1]$ defined by $([2] \to [1], [2] \to [1]) = ((0,1 \mapsto 0; 2 \mapsto 1), (0 \mapsto 0; 1,2 \mapsto 1))$ in $\mathbf{\Delta}$. Triangle (3.1.7) is given by $((0 \mapsto 0; 1,2 \mapsto 1), (0,1 \mapsto 0; 2 \mapsto 1))$. The hypotenuse in both triangles is the edge $\left([1] \xrightarrow{\mathrm{id}} [1], [1] \xrightarrow{\mathrm{id}} [1]\right) \in (\Delta[1] \times \Delta[1])_1$.

Construction 3.1.8 (exit shuffles)**.** Any pair $1 \leq j \leq k$ of natural numbers determines a $(1, k-1)$-shuffle $\mathcal{S}_j^k = \mathcal{S}_j \colon \Delta[k] \to \Delta[1] \times \Delta[k-1]$ by setting

$$\mathcal{S}_j = \begin{cases} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1_j & 1_{j+1} & 1 & \cdots & 1 \\ 0 & 1 & \cdots & j-1 & j-1 & j & j+1 & \cdots & k-1 \end{bmatrix}, & j < k \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 2 & \cdots & k-2 & k-1 & k-1 \end{bmatrix}, & j = k \end{cases}$$

in path notation, where the subscript j indicates the column number, with column count starting at 0.

This is the non-degenerate element of $(\Delta[1] \times \Delta[k-1])_k$ induced in $\mathbf{\Delta}$ by the poset map

$$[k] \to [1] \times [k-1]$$

given by

$$i \mapsto \begin{cases} (0, i), & i < j \\ (1, i-1), & i \geq j. \end{cases}$$

We call $\mathcal{S}_j$ an *exit shuffle*, and j its *exit index*.[2] It has multiple left inverses, but we will use a particular one, $\mathcal{C}_j^k = \mathcal{C}_j$, defined to be postcomposition with the poset map

$$[1] \times [k-1] \to [k]$$

given by

$$(0, i) \mapsto \begin{cases} i, & i < j \\ j-1, & i \geq j \end{cases}, \quad (1, i) \mapsto \begin{cases} j, & i < j \\ i+1, & i \geq j \end{cases}.$$

This choice for $\mathcal{C}$ is justified by results below such as Lemmas 3.1.13 and 3.1.14.

Definition 3.1.9. Let $\iota \colon \mathcal{L} \to \mathcal{N}$ be a map of simplicial sets. For $k \geq 1$, we define

$$\mathcal{P}_{k-1}^{\Delta} \subset \mathcal{N}_k \times \{1, \ldots, k\}$$

to be the subset consisting of pairs (γ, j) such that in the diagram

$$\begin{tikzcd} \Delta[k] \arrow[rr, "\gamma"] \arrow[dr, bend right, "\mathcal{S}_j"'] & & \mathcal{N} \\ & \Delta[1] \times \Delta[k-1] \arrow[ul, two heads, bend right, "\mathcal{C}_j"'] \arrow[ur, dashed, "\Gamma=\gamma\circ\mathcal{C}_j"'] & \end{tikzcd}$$

the arrow Γ lifts to the mapping cocylinder, i.e., it is in the image of the natural map $\mathcal{P} \to \mathcal{N}^{\Delta[1]}$. We call a pair $(\gamma, j) \in \mathcal{P}_*^{\Delta}$ an *exit path of index* j.

One can think of $\Gamma = \gamma \circ \mathcal{C}_j$ as a scaffolding around γ that gives it shape and direction. We will differentiate its three parts – its base, its ladders, and its top – as they relate to γ in Definition 3.1.11.

[2]Exit shuffles are exactly the non-degerate simplices of $\Delta[1] \times \Delta[k]$ of maximal dimension, $k+1$.

Remark 3.1.10 (exit indices at depth 1)**.** In terms of ordinary stratified geometry, Construction 3.1.8 corresponds to the following phenomenon: a stratified k-simplex or k-chain $\Delta^k \to X$ of X is a map of stratified spaces, where $\Delta^k = \overline{C}^k(\mathrm{pt})$ is the k-fold closed cone on the point. The closed cone $\overline{C}(Y)$ of a stratified space $Y \to \mathcal{P} = \mathcal{P}_Y$, where $\mathcal{P}$ is the stratifying poset (equipped with the Alexandrov topology so that downward-closed subsets are closed) has

$$\mathrm{pt} \coprod_{\{0\}\times Y} [0,1] \times Y$$

as its underlying space, and

$$\mathcal{P}_{\overline{C}(Y)} = \mathcal{P}_Y^{\triangleleft},$$

i.e., $\mathcal{P}_Y$ with a minimal element adjoined, as its stratifying poset, together with the obvious stratification $\overline{C}(Y) \to \mathcal{P}_Y^{\triangleleft}$. Now, the stratified map $\Delta^k \to X$ comes with a commutative topological square

$$\begin{tikzcd} \Delta^k \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathcal{P}_{\Delta^k} \arrow[r, "s_f"'] & \mathcal{P}_X \end{tikzcd}.$$

Clearly we have $\mathcal{P}_{\Delta^k} \simeq [k]$ as posets. If

$$P_X \simeq \{a \prec b\},$$

then the poset map s_f is determined by a unique minimal 'exit index' $j \in [k]$. Namely, let $j = 0$ if s_f is constant, or else let j be the smallest number such that

$$s_f(j-1 \prec j) = a \prec b,$$

referring to s_f applied to an arrow. This is well-defined since $[k]$ is connected. As we do not refer to stratified paths explicitly, however, the different levels (indices) at which a path may exit (from the stratum X_a) give for us different sorts of non-invertible paths. Note also that we do not consider exit shuffles of index 0, as the corresponding k-chains are competely contained within the smooth manifold X_b, and similarly we do not consider '$j = k+1$', i.e., paths contained within X_a. (Besides, these indices do not determine shuffles in the ordinary sense.) This analogy suggests a natural, albeit notationally heavy, generalisation, using multiple exit indices, of the depth-1 Construction 3.1.8 to higher depth, but we will not pursue this here.

The aim of Definition 3.1.9 is three-fold.

- It helps group elements of $\mathcal{P}_*^{\Delta}$ into three classes (Definition 3.1.11), which will play different roles.
- It 'fixes orientation', in the sense that the faces of γ that touch $\mathcal{L}$ are directed *away* from $\mathcal{L}$ due to the orientation of the accompanying $\Gamma \in \mathcal{P}_*$. This precludes 'paths from $\mathcal{N}$ to $\mathcal{M}$', a.k.a. enter paths. The orientation depends on the exit index, so:
- Unequal pairs $(\gamma, j) \neq (\gamma, j') \in \mathcal{P}_*^{\Delta}$ that share their first coordinate play different roles, and this is indispensable.

Definition 3.1.11. Let $k \geq 1$, $(\gamma, j) \in \mathcal{P}^{\Delta}_{k-1}$, and let $d_i = \partial_i^*$ be a face map. Then $d_i(\gamma)$ is either

- or *low* if it factors as follows:

$$\begin{tikzcd} \Delta[k-1] \arrow[r, hook, "\partial_i"] \arrow[dr, dotted, "\exists"'] & \Delta[k] \arrow[rr, "\gamma"] \arrow[dr, "\mathcal{S}_j"] & & \mathcal{N} \\ & \{0\} \times \Delta[k-1] \arrow[r, hook] & \Delta[1] \times \Delta[k-1] \arrow[ur, "\Gamma=\gamma\circ\mathcal{C}_j"'] & \end{tikzcd};$$

- *vertical* if it does not factor as follows:

$$\begin{tikzcd} \Delta[k-1] \arrow[r, hook, "\partial_i"] \arrow[dr, dotted, "\nexists"'] & \Delta[k] \arrow[dr, "\mathcal{S}_j"] & \\ & (\{0\} \amalg \{1\}) \times \Delta[k-1] \arrow[r, hook] & \Delta[1] \times \Delta[k-1] \end{tikzcd};$$

- or *upper* if it factors as follows:

$$\begin{tikzcd} \Delta[k-1] \arrow[r, hook, "\partial_i"] \arrow[dr, dotted, "\exists"'] & \Delta[k] \arrow[dr, "\mathcal{S}_j"] & \\ & \{1\} \times \Delta[k-1] \arrow[r, hook] & \Delta[1] \times \Delta[k-1] \end{tikzcd}.$$

In the exit path ∞-category of Definition 3.2.2 below, vertical paths will remain non-invertible, low faces will become simplices in $\mathcal{M}$, and upper faces in $\mathcal{N}$. Writing '$d_i(\gamma)$ is vertical', etc., is slightly abusive, since whether a face is vertical, low or upper depends on (and in fact only on) the exit index. This should not cause any confusion because we do not use these adjectives in any other context. We have adopted 'low' and 'upper' from [32], where they were used in a similar context.

Definition 3.1.12. Let $k \geq 1$. For ∂_i and $\mathcal{S}_j$ as in Definition 3.1.11, and for σ_i a degeneracy, we write

$$\flat^k_{j,i} = \flat_{j,i} \in [k-1] \text{ (resp. } \sharp^k_{j,i} = \sharp_{j,i} \in [k])$$

for the smallest number whose image under

$$\mathcal{S}_j\partial_i \colon [k-1] \to [1] \times [k-1] \text{ (resp. under } \mathcal{S}_j\sigma_i \colon [k+1] \to [1] \times [k-1])$$

has first coordinate 1. We leave $\flat^k_{k,k}$ undefined.

For instance, for $k = 5$, $j = 2$, $i \geq 2$, we have $\flat = 2$, but for $i < 2$ (with k, j unchanged), we have $\flat = 1$; in general $\flat \in \{j, j-1\}$, depending on the relative positions of i and j in $\{0, \ldots, k\}$. We will note explicit formulas for $\flat$ and $\sharp$ in the proof of Lemma 3.2.3: see (3.2.6) and (3.2.10). Their derivation is left as an exercise.

Lemma 3.1.13.

- *Let $k \geq 2$ and assume $(j, i) \neq (k, k)$. The composition*

$$\Delta[1] \times \Delta[k-2] \xrightarrow{\mathcal{C}_\flat} \Delta[k-1] \xrightarrow{\partial_i} \Delta[k] \xrightarrow{\mathcal{S}_j} \Delta[1] \times \Delta[k-1],$$

where $\flat = \flat^k_{j,i}$, preserves the first coordinate.

- *The composition*

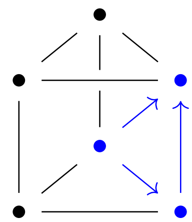
$$\Delta[1] \times \Delta[k] \xrightarrow{\mathcal{C}_\sharp} \Delta[k+1] \xrightarrow{\sigma_i} \Delta[k] \xrightarrow{\mathcal{S}_j} \Delta[1] \times \Delta[k-1],$$

where $\sharp = \sharp^k_{j,i}$, preserves the first coordinate.

Proof. This is a direct check. ☐

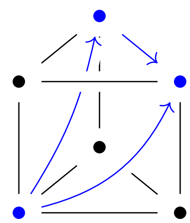
Lemma 3.1.14. *Let $k \geq 2$. If $(\gamma, j) \in \mathcal{P}^\Delta_{k-1}$ and $d_i(\gamma)$ is vertical, then $d_i(\gamma, j) := \big(d_i\gamma, \flat^k_{j,i}\big) \in \mathcal{P}^\Delta_{k-2}$.*

To illustrate, for $k = 3$,



(3.1.15)

is a vertical face of exit index $\flat = 2 = j - 1$, where $(\gamma, 3)$ itself, the 'lower right' tetrahedron, is omitted. Similarly,



(3.1.16)

is a vertical face of index $\flat = 1 = j$, where $(\gamma, 1)$ is the upper left tetrahedron.[3]

Proof of Lemma 3.1.14. It suffices to consider the diagram

$$\begin{tikzcd}
\Delta[k-1] \arrow[r, hook, "\partial_i"] \arrow[ddrr, hook, bend right, "\mathcal{S}_\flat"'] & \Delta[k] \arrow[rr, "\gamma"] \arrow[dr, hook, bend right, "\mathcal{S}_j"'] & & \mathcal{N} \\
& & \Delta[1] \times \Delta[k-1] \arrow[ul, two heads, bend right, "\mathcal{C}_j"'] \arrow[ur, dashed, "\Gamma"] & \\
& & \Delta[1] \times \Delta[k-2] \arrow[u, dashed, "d'"] \arrow[uull, two heads, bend right, "\mathcal{C}_\flat"'] \arrow[uur, dashed, bend right, "\Gamma'"'] &
\end{tikzcd}, \qquad (3.1.17)$$

which commutes by construction. Lemma 3.1.13 implies in particular that the restriction of $d' = \mathcal{S}_i\partial_i\mathcal{C}$ to $\{0\} \times \Delta[k-2]$ factors through $\{0\} \times \Delta[k-1]$, which implies that $\Gamma' = \Gamma d'$ lifts to $\mathcal{P}_{k-2}$, as desired. Note that the case $j = i = k$ is precluded by verticality. ☐

[3]In these pictures, the boundary triangles of the prisms are oriented clockwise.

Remark 3.1.18. Lemma 3.1.14 does not promote to an if-and-only-if statement. Low faces also descend to $\mathcal{P}^{\Delta}_{k-2}$, but in a different way. Upper faces may or may not. These facts will play no role below.

We close this section by noting the completely analogous fact for degeneracies.

Lemma 3.1.19. *Let $k \geq 1$. If $(\gamma, j) \in \mathcal{P}^{\Delta}_{k-1}$, then $s_i(\gamma, j) \coloneqq \left(s_i\gamma, \sharp^k_{j,i}\right) \in \mathcal{P}^{\Delta}_k$.*

3.2. Exit paths

Remark 3.2.1. If $\iota\colon \mathcal{L} \hookrightarrow \mathcal{N}$ is a cofibration, then an exit path $(\gamma, j) \in \mathcal{P}^{\Delta}_{k-1}$ determines a canonical $(k-1)$-simplex $\Delta[k-1] \to \mathcal{L}$ of $\mathcal{L}$, namely (recall Definition 3.1.9) the restriction of $\Gamma = \gamma \circ \mathcal{C}_j$ along $\{0\} \times \Delta[k-1] \hookrightarrow \Delta[1] \times \Delta[k-1]$ factors then *uniquely* through $\mathcal{L}$.

We are now ready to give one of the main constructions of this dissertation.

Definition 3.2.2. Let a span

$$\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xhookrightarrow{\iota} \mathcal{N}\right)$$

of simplicial sets be given, where ι is a cofibration. We define a new simplicial set, $\mathcal{EX} = \mathcal{EX}(\mathfrak{S})$, as follows:

- $\mathcal{EX}_0 = \mathcal{M}_0 \amalg \mathcal{N}_0$.
- $\mathcal{EX}_k = \mathcal{M}_k \amalg \mathcal{P}^{\Delta}_{k-1} \amalg \mathcal{N}_k$ for $k \geq 1$.
- Face and degeneracy maps restricted to $\mathcal{M}_k$ and $\mathcal{N}_k$ are those of $\mathcal{M}$ and $\mathcal{N}$.
- For $k = 1$ and $\gamma = (\gamma, 1) \in \mathcal{P}^{\Delta}_0 \subset \mathcal{N}_1$, we set[4]
$$d_1(\gamma, 1) = \pi(d_1\gamma) \in \mathcal{M}_0,$$
$$d_0(\gamma, 1) = \tau(d_0\gamma) \in \mathcal{N}_0.$$
- For $k \geq 2$, $(\gamma, j) \in \mathcal{P}^{\Delta}_{k-1}$, and d_i a face map:
  - if $d_i\gamma$ is vertical,[5] then we set $d_i(\gamma, j) = \left(d_i\gamma, \flat_{j,i} \in \mathcal{P}^{\Delta}_{k-2}\right)$.[6]
  - if $d_i\gamma$ is low, then we set $d_i(\gamma, j) = \pi(d_i\gamma) \in \mathcal{M}_{k-1}$.
  - if $d_i\gamma$ is upper, then we set $d_i(\gamma, j) = \tau(d_i\gamma) \in \mathcal{N}_{k-1}$.
- For $k \geq 1$, $(\gamma, j) \in \mathcal{P}^{\Delta}_{k-1}$, and s_i a degeneracy: $s_i(\gamma, j) \coloneqq (s_i\gamma, \sharp_{j,i}) \in \mathcal{P}^{\Delta}_k$.[7]

Lemma 3.2.3. *$\mathcal{EX}(\mathfrak{S})$ is a simplicial set.*

Proof. We will verify the simplicial identities. Below, we assume $k \geq 2$ or $k \geq 3$ depending on applicability, and that $(\gamma, e) \in \mathcal{P}^{\Delta}_{k-1}$. For completeness, we have included a proof for the case $k = 1$ at the end, though it is better considered an exercise.

[4](noting $\mathcal{S} = \mathrm{id}$, $\mathcal{C} = \mathrm{id}$ if $k = 1$ (Construction 3.1.8), and using Remarks 3.1.3 and 3.2.1)
[5](Definition 3.1.11)
[6](Lemma 3.1.14)
[7](Lemma 3.1.19)

$d_id_j = d_{j-1}d_i$ for $i < j$: We start by showing that

$$\flat^{k-1}_{\flat^k_{e,j},i} = \flat^{k-1}_{\flat^k_{e,i},j-1}. \tag{3.2.4}$$

It helps to distinguish the cases

$$(1)\ e \le i < j,\ (2)\ i < e \le j,\ \text{and}\ (3)\ i < j < e. \tag{3.2.5}$$

We have (by a direct check)

$$\flat^k_{e,j} = \begin{cases} e, & j \ge e \\ e-1, & j < e \end{cases} \tag{3.2.6}$$

and thus if (1), then $\mathrm{L} := \flat^{k-1}_{\flat^k_{e,j},i} = \flat^{k-1}_{e,i} = e$ and $\mathrm{R} := \flat^{k-1}_{\flat^k_{e,i},j-1} = \flat^{k-1}_{e,j-1} = e$. If (2), then $\mathrm{L} = \flat^{k-1}_{e,i} = e-1$ and $\mathrm{R} = \flat^{k-1}_{e-1,j-1} = e-1$. Finally, if (3), then $\mathrm{L} = \flat^{k-1}_{e-1,i} = e-2$ and $\mathrm{R} = \flat^{k-1}_{e-1,j-1} = e-2$. We should note that in the case (2), e is at least 1, and in (3) it is at least 2, so that the expressions make sense.

This finishes the verification if all involved faces of (γ, e) are vertical. Otherwise, Lemma 3.1.13 and Diagram (3.1.17) imply the statement; in any of the cases where the case excluded in Lemma 3.1.13 is involved, the face in question is low. We will give this argument here once and will not repeat it in the verification of the other simplicial identities below.

Consider the diagram

$$\begin{tikzcd}
\Delta[k-2] \arrow[r, hook, "\partial_i"] \arrow[d, bend right, "\mathcal{S}_{\flat'}"'] & \Delta[k-1] \arrow[r, hook, "\partial_j"] \arrow[d, bend right, "\mathcal{S}_{\flat}"'] & \Delta[k] \arrow[r, "\gamma"] \arrow[d, bend right, "\mathcal{S}_{e}"'] & \mathcal{N} \\
\Delta[1] \times \Delta[k-3] \arrow[r, dashed] \arrow[u, bend right, "\mathcal{C}_{\flat'}"'] & \Delta[1] \times \Delta[k-2] \arrow[r, dashed] \arrow[u, bend right, "\mathcal{C}_{\flat}"'] & \Delta[1] \times \Delta[k-1] \arrow[u, bend right, "\mathcal{C}_{e}"'] \arrow[ur, dashed] &
\end{tikzcd}. \tag{3.2.7}$$

Without loss of generality, say $d_i(d_j(\gamma)) = (\partial_j\partial_i)^*\gamma$ is low, so we need to show that so is $d_{j-1}d_i(\gamma)$. That $\mathcal{S}_\flat\partial_i$ factors through $\{0\} \times \Delta[k-2]$ is equivalent to $\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i$ factoring thusly by Lemma 3.1.13. Now, $\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i = \mathcal{S}_e\partial_j\partial_i$ by the construction of $\mathcal{C}_\flat$, and similarly $\mathcal{S}_e\partial_j\partial_i = \mathcal{S}_e\partial_j\partial_i\mathcal{C}_{\flat'}\mathcal{S}_{\flat'}$. Together with the same calculation for ∂_i and ∂_j replaced respectively by ∂_{j-1} and ∂_i in Diagram (3.2.7), we see that

$$\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i = \mathcal{S}_e\partial_j\partial_i\mathcal{C}_{\flat'}\mathcal{S}_{\flat'} \quad \text{and} \quad \mathcal{S}_e\partial_i\mathcal{C}_\flat\mathcal{S}_\flat\partial_{j-1} = \mathcal{S}_e\partial_i\partial_{j-1}\mathcal{C}_{\flat'}\mathcal{S}_{\flat'}. \tag{3.2.8}$$

The indices $\flat^{(\prime)}$ in the two equations are a priori *not* the same (as they are calculated for different pairs of indices themselves), but we just showed above in Equation (3.2.4) that the primed flats on the right hand sides do coincide. Combined with the same simplicial identity for $\mathcal{N}$, this means that the right hand sides in (3.2.8) agree, which implies the statement.

$d_is_j = s_{j-1}d_i$ for $i < j$: Similarly, we first show

$$\mathrm{L} = \flat^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\flat^k_{e,i},j-1} = \mathrm{R}, \tag{3.2.9}$$

using the cases (1)–(3) from (3.2.5). Note that

$$\sharp^k_{e,j} = \begin{cases} e, & j \geq e \\ e+1, & j < e \end{cases} \tag{3.2.10}$$

which, together with (3.2.6), implies that if (1), then $\mathrm{L} = \flat^{k-1}_{e,i} = e$ and $\mathrm{R} = \sharp^{k-1}_{e,j-1} = e$. If (2), then $\mathrm{L} = \flat^{k-1}_{e,i} = e-1$ and $\mathrm{R} = \sharp^{k-1}_{e-1,j-1} = e-1$. Finally, if (3), then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e$ and $\mathrm{R} = \sharp^{k-1}_{e-1,j-1} = e$. Now, Lemma 3.1.13 and Diagrams (3.1.17) and (3.2.7) (mutatis mutandis; e.g., using (3.2.9) instead of (3.2.4) for (3.2.8)) again finish the verification, analogously to the above. We no longer mention this below.

$\underline{d_i s_j = \mathrm{id} \text{ for } i = j \text{ or } i = j+1:}$ We show

$$\mathrm{L} = \flat^{k-1}_{\sharp^k_{e,j},i} = e.$$

If $e \leq j$, then $\mathrm{L} = \flat^{k-1}_{e,i} = e$. If $i = j$ and $j < e$, then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e$. If $i = j+1$ and $e \geq i$, then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e$. This covers all cases.

$\underline{d_i s_j = s_j d_{i-1} \text{ for } i > j+1:}$ We show

$$\mathrm{L} = \flat^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\flat^k_{e,i-1},j} = \mathrm{R}.$$

If $e \leq j$, then $\mathrm{L} = \flat^{k-1}_{e,i} = e = \sharp^{k-1}_{e,j} = \mathrm{R}$. If $j+1 \leq e < i-1$, then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e = \sharp^{k-1}_{e-1,j} = \mathrm{R}$. If $e = i-1$, then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e+1 = \sharp^{k-1}_{e,j} = \mathrm{R}$. If $e = i$, then $\mathrm{L} = \flat^{k-1}_{e+1,i} = e = \sharp^{k-1}_{e-1,j} = \mathrm{R}$. Finally, if $e > i$, both sides are again equal to e.

$\underline{s_i s_j = s_{j+1} s_i \text{ for } i \leq j:}$ Finally, we show

$$\mathrm{L} = \sharp^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\sharp^k_{e,i},j+1} = \mathrm{R}.$$

Similarly to the first identity above, it helps to distinguish the cases

$$(1)\ e \leq i \leq j,\ (2)\ i < e \leq j, \text{ and } (3)\ i \leq j < e.$$

If (1), then $\mathrm{L} = \sharp^{k-1}_{e,i} = e = \sharp^{k-1}_{e,j+1} = \mathrm{R}$. If (2), then $\mathrm{L} = \sharp^{k-1}_{e,i} = e+1 = \sharp^{k-1}_{e+1,j+1} = \mathrm{R}$. If (3), then $\mathrm{L} = \sharp^{k-1}_{e+1,i} = e+2 = \sharp^{k-1}_{e+1,j+1} = \mathrm{R}$.

$\underline{\text{The case } k = 1:}$ Here, we necessarily have $e = 1$. The first simplicial identity (in the order presented above) is not applicable for dimension reasons. For the second, the only applicable case is $i = 0$, $j = 1$. Then we have $d_0 s_1(\gamma, 1) = d_0(s_1\gamma, \sharp_{1,1}) = d_0(s_1\gamma, 1)$. This face is upper, so

$$d_0 s_1(\gamma, 1) = d_0 s_1 \gamma = s_0 d_0 \gamma.$$

On the other hand, $d_0(\gamma, 1)$ is also upper, so

$$s_0 d_0(\gamma, 1) = s_0 d_0 \gamma$$

as well. For the third identity, the only applicable cases are $i = 0, 1, 2$. If $i = 0$, then $d_0 s_0(\gamma, 1) = d_0(s_0\gamma, \sharp_{1,0}) = d_0(s_0\gamma, 2)$. This face is vertical, so

$$d_0 s_0(\gamma, 1) = (d_0 s_0, \flat_{2,0}) = (d_0 s_0 \gamma, 1) = (\gamma, 1),$$

as desired. If $i = 1$, similarly $d_1 s_0(\gamma, 1) = (d_1 s_0 \gamma, \flat_{2,1}) = (\gamma, 1)$ by verticality. If $i = 2$, again $d_2 s_1(\gamma, 1) = (d_2 s_1 \gamma, \flat_{\sharp 1,1,2}) = (\gamma, 1)$ by verticality. For the fourth identity, the only applicape case is $i = 2$, $j = 0$. Then, on one hand, we have $d_2 s_0(\gamma, 1) = d_2(s_0\gamma, 2) = \pi(d_2 s_0 \gamma)$ since the d_2-face is here low, and on the other hand, we have that $d_1(\gamma, 1)$ is also low, so

$$s_0 d_1(\gamma, 1) = s_0 \pi(d_1\gamma) = \pi(s_0 d_1 \gamma) = \pi(d_2 s_0 \gamma),$$

as desired. The fifth and last identity accepts the general treatment we gave above, since at each step the simplex remains vertical by construction. □

Theorem 3.2.11. *If $\mathcal{M}, \mathcal{L}, \mathcal{N}$ are ∞-categories, $\pi\colon \mathcal{L} \to \mathcal{M}$ is a right fibration, and $\iota\colon \mathcal{L} \to \mathcal{N}$ is a cofibration, then $\mathcal{EX}\left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ is an ∞-category.*

Proof. We directly check the weak Kan property, first giving a verbose proof for inner 2- and 3-horns before the general case, which is analogous. The main idea is that given a horn with non-invertible faces, we can lift those in $\mathcal{M}$ to $\mathcal{N}$ along π and take a filler therein, which, coupled with an appropriate exit index, lifts the original horn. We will sometimes not distinguish $\mathcal{L}$ from its image $\iota(\mathcal{L})$ in notation.

<u>2-horns.</u> Let $h\colon \Lambda^2_1 \to \mathcal{EX}$ be given. The only two non-trivial cases occur when at least one of the edges

$$h|_{ij}\colon \{i < j\} = \Delta[1] \hookrightarrow \Lambda^2_1 \xrightarrow{h} \mathcal{EX}$$

that constitute the horn lies in $\mathcal{P}^\Delta_0$. (See Footnote 8 concerning the notation.)

(1) First, say
$$h|_{01} = (h_{01}, 1) \in \mathcal{P}^\Delta_0.$$
Then we have $h|_{12} \in \mathcal{N}_1$ as by construction the endpoint $d_0(h|_{01}) = \tau(d_0 h_{01})$, which must be the initial point of $h|_{12}$, lies in $\mathcal{N}_0$. Now the horn
$$h_{01} \cup h|_{12}\colon \Lambda^2_1 \to \mathcal{N}$$
has a filler $H\colon \Delta[2] \to \mathcal{N}$. But then $(H, 1) \in \mathcal{P}^\Delta_1$ fills h: the composition $\mathcal{S}_1 \circ \partial_0\colon [1] \hookrightarrow [2] \hookrightarrow [1] \times [1]$ is $0 \mapsto 1 \mapsto (1, 0)$, $1 \mapsto 2 \mapsto (1, 1)$ which means $d_0 H$ is upper, so that
$$d_0(H, 1) = d_0 H = h|_{12} \in \mathcal{N}_1.$$
Similarly, $\mathcal{S}_1 \circ \partial_2(0) = \mathcal{S}_1(0) = (0, 0)$, $\mathcal{S}_1 \circ \partial_2(1) = \mathcal{S}_1(1) = (1, 0)$ so that $d_2 H$ is vertical, yielding
$$d_2(H, 1) = (d_2 H, \flat_{1,2}) = (h_{01}, 1) \in \mathcal{P}^\Delta_0$$
using (3.2.6). This shows that $(H, 1)$ is a filler for h.

(2) Let us assume, more interestingly, that
$$h|_{12} = (h_{12}, 1) \in \mathcal{P}^\Delta_0.$$
Necessarily, $h|_{01} \in \mathcal{M}_1$, as the initial point $d_1(h|_{12}) = \pi(d_1 h_{12})$, which must coincide with the endpoint of $h|_{01}$, is by construction in $\mathcal{M}_0$.

The induced lifting problem

$$\begin{tikzcd} \{1\} = \Lambda^1_0 \arrow[r, "d_1 h_{12}"] \arrow[d, hook] & \mathcal{L} \arrow[d, "\pi"] \\ \Delta[1] \arrow[r, "h|_{01}"'] \arrow[ur, dotted, "H_{01}"] & \mathcal{M} \end{tikzcd}$$

admits by assumption a solution H_{01}. We thus have an induced horn

$$\iota(H_{01}) \cup h_{12} \colon \Lambda^2_1 \to \mathcal{N}$$

with a filler $H \in \mathcal{N}_2$. But now $(H, 2) \in \mathcal{P}^{\Delta}_1$ fills h: we have that $d_0 H$ is vertical as $\mathcal{S}_2 \circ \partial_0 \colon [1] \to [1] \times [1]$ sends $0 \mapsto (0,1)$; $1 \mapsto (1,1)$, so

$$d_0(H,2) = (d_0 H, \flat_{2,0}) = (h_{12}, 1) \in \mathcal{P}^{\Delta}_0.$$

Similarly, $\mathcal{S}_2 \circ \partial_2 \colon 0 \mapsto (0,0);\ 1 \mapsto (0,1)$ means $d_2 H$ is low and so

$$d_2(H,2) = \pi(d_2 H) = \pi(H_{01}) = h|_{01} \in \mathcal{M}_1.$$

This shows that $(H, 2)$ is a filler for h.

<u>3-horns.</u> Let first

$$h \colon \Lambda^3_1 \to \mathcal{EX}$$

be given, which misses the 023-face. The non-trivial cases to check occur when h is not wholly contained within $\mathcal{M}$ or $\mathcal{N}$. Suppose

(1) that the 013-face

$$h|_{013} \colon \Delta[2] = \{0 < 1 < 3\} \hookrightarrow \Lambda^3_1 \to \mathcal{EX}$$

is in $\mathcal{M}_2$.[8] Then if any other non-degenerate sub-2-simplex of h is also low,[9] so must all others, which would yield a non-case as h would lie entirely within $\mathcal{M}$. But since no other sub-2-simplex of h can be upper while $h|_{013}$ is low, we may assume that *all other* non-degenerate sub-2-simplices of h are vertical. Now, $h|_{123} = (h, e') \in \mathcal{P}^{\Delta}_1$ must be vertical with the 03-edge, common with the assumed low face $h|_{013}$, itself necessarily low. But then the vertex $h|_2 \in \mathcal{N}_0$ must be upper, which is absurd since there is no exit index $e' \in \{1, 2\}$ such that the exit shuffle $\mathcal{S}_{e'} \colon [2] \to [1] \times [1]$ sends $0, 3$ to $\{0\} \times [1]$ while simultaneously sending 2 to $\{1\} \times [1]$: $2 < 3$ in $[2]$ implies $\mathcal{S}_{e'}(2) < \mathcal{S}_{e'}(3)$ in $[1] \times [1]$. We conclude that $h|_{013}$ cannot be in $\mathcal{M}_2$ if h is not already wholly within $\mathcal{M}$. *So, this is a non-case.*

(2) that the 123-face

$$h|_{123} \colon \Delta[2] = \{1 < 2 < 3\} \hookrightarrow \Lambda^3_1 \to \mathcal{EX}$$

[8] We will continue using slightly abusive notation like $\Delta[2] = \{0 < 1 < 3\}$ which is similar to $\Delta[2] = \Delta\{0 < 1 < 3\}$ or $\Delta^{\{0<1<3\}}$ since it is suggestive, commonplace, and should cause no confusion.

[9] We call a sub-simplex $(\Delta[\ell] \hookrightarrow \Lambda^k_i \to \mathcal{EX}) \in \mathcal{EX}_{k-1}$, $1 \leq \ell < k$ of a horn *low* if it is in $\mathcal{M}_\ell$, *vertical* if in $\mathcal{P}^{\Delta}_{\ell-1}$, and *upper* if in $\mathcal{N}_\ell$. Similarly when $\ell = 0$ with *low/upper*.

is in $\mathcal{M}_2$. All other 2-faces being vertical similarly implies that the vertex $h|_0$ is upper, which gives a contradiction in the same way. We conclude that *this is also a non-case.*

(3) that the 012-face

$$h|_{012}\colon \Delta[2] = \{0 < 1 < 2\} \hookrightarrow \Lambda^3_1 \to \mathcal{EX}$$

is in $\mathcal{M}_2$. We may similarly assume all other 2-faces are vertical, and so in particular $h|_3$ upper, which here does not give a contradiction as $3 \in \Lambda^3_1 \subset \Delta[3]$ is final. We obtain that $h|_{013} = (h_{013}, 2), h|_{123} = (h_{123}, 2) \in \mathcal{P}^\Delta_1$ both have exit index 2, as they have a single low edge each (the 01- and 12-edges, respectively). Now, we have an induced horn in $\mathcal{L}$ given by $h|_{01} \cup h|_{12}\colon \Lambda^2_1 \to \mathcal{L}$, which constitutes the lifting problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "h_{01}\cup h_{12}"] \arrow[d, hook] & \mathcal{L} \arrow[d, "\pi"] \\
\Delta[2] \arrow[r, "h|_{012}"'] \arrow[ur, dotted, "H_{012}"] & \mathcal{M}
\end{tikzcd}$$

which admits a solution H_{012}. This yields an induced horn

$$H_{012} \cup h_{013} \cup h_{123}\colon \Lambda^3_1 \to \mathcal{N}$$

which admits a filler H. Now, $(H, 3) \in \mathcal{P}^\Delta_2$ fills h: the face d_3H is low since

$$\mathcal{S}_3 \circ \partial_3\colon [2] \hookrightarrow [3] \hookrightarrow [1] \times [2]$$

sends $i \mapsto (0, i)$, which implies

$$d_3(H, 3) = \pi(d_3H) = \pi(H_{012}) = h|_{012},$$

as desired. The remaining faces d_iH, $i \neq 3$, are vertical since $\mathcal{S}_3 \circ \partial_i$ sends $0 \mapsto (0, 0)$ while $3 \mapsto (1, 2)$. By construction we have

$$d_0(H, 3) = (d_0H, \flat_{3,0}) = (h_{123}, 2) = h|_{123}$$

and similarly $d_1(H, 3) = h|_{123}$, $d_2(H, 3) = h|_{013}$, using $\flat_{3,i} = 2$, as desired.

(4) that h has an upper 2-face. Similarly to the above, we may argue that it cannot have a low 2-face, and if it had any *other* upper 2-face, it would be contained entirely within $\mathcal{N}$ where the lifting problem is trivial, and so we may assume that the remaining 2-faces are all vertical. Again similarly to the above, we have that the only non-non-case is when the 123-face

$$h|_{123}\colon \Delta[2] = \{1 < 2 < 3\} \hookrightarrow \Lambda^3_1 \to \mathcal{EX}$$

is upper as the other cases contradict the partial order on $[1] \times [2]$; in particular, the vertex $h|_0$ is low. Moreover, any vertical 2-face of h must have exit index 1 for otherwise it would have a low edge, which is impossible as there is only a single low vertex. Now, h is in this case given by a horn

$$h_{012} \cup h_{013} \cup h_{123} = \widetilde{h}\colon \Lambda^3_1 \to \mathcal{N}$$

with $\widetilde{h}|_0$ in $\iota(\mathcal{L})$. Taking a filler H of $\widetilde{h}$ in $\mathcal{N}$, we see that $(H,1)$ fills h: the faces $d_{1/2/3}H$ are vertical since $\mathcal{S}_1\colon [3] \to [1] \times [2]$ sends $0 \mapsto (0,0)$; $1 \leq i \mapsto (1, i-1)$, which means

$$d_i(H,1) = (d_iH, \flat_{1,i}) = (d_iH, 1),$$

for $i \geq 1$, as desired. On the other hand, $\mathcal{S}_1 \circ \partial_0$ has image inside $\{1\} \times [2]$, so d_0H is top. We obtain

$$d_0(H,1) = \tau(d_0H) = h|_{123} \in \mathcal{N}_2,$$

as desired.

(5) finally that all 2-faces of h are vertical. Let us rule out a few possibilities by yet more pigeon-holing arguments: the presence of three (out of the four in total) low resp. upper vertices implies that there is a low resp. upper face, which means we must have two low and two upper vertices each. Now, $h|_0$ and $h|_1$ must be low, and $h|_2$ and $h|_3$ upper. For if $h|_0$ were upper and $h|_i$ $(i \geq 1)$ low, the edge $h|_{0i}$ would be a path $\mathcal{N} \to \mathcal{M}$, which is excluded by construction, and similarly if $h|_1$ were upper, taking $i \geq 2$. Therefore, the 2-faces

$$h|_{012},\ h|_{013},$$

namely those that contain both $h|_0$ and $h|_1$, have exit index 2, while those which contain only one low vertex have index 1. Of latter type there is only one:

$$h|_{123}.$$

Although $h|_{023}$ is missing, it would have had to have index 1 by the same argument. Therefore, adopting the notation from Case (4), we may take a lift $H\colon \Delta[3] \to \mathcal{N}$ of $\widetilde{h}\colon \Lambda^3_1 \to \mathcal{N}$ such that the restriction to $\{0\} \times \Delta[2]$ of the composition $H \circ \mathcal{C}_2\colon \Delta[1] \times \Delta[2] \to \mathcal{N}$ still factors through $\mathcal{L}$, independently of the choice of H. Indeed, $(H,2) \in \mathcal{P}^{\Delta}_2$ fills h: since $\flat_{2,2/3} = 2$ and $\flat_{2,0/1} = 1$, we have $d_{2/3}(H,2) = h|_{013/012}$, $d_0(H,2) = h|_{123}$. Note also that the exit index being 2 excludes low or upper faces in this dimension.

Let now

$$h\colon \Lambda^3_2 \to \mathcal{EX}$$

be given, which misses the 013-face. The non-trivial cases occur when

(1) h has a low face. As in the case of a 1st 3-horn, we may exclude all cases except the one where the low face is $h|_{012} \in \mathcal{M}_2$ and all other 2-faces are again necessarily vertical, with sole upper vertex $h|_3$. Now, the faces $h|_{123}, h|_{023} \in \mathcal{P}^{\Delta}_1$ necessarily have exit index 2, $h|_{ijk} = (h_{ijk}, 2)$, and their source edges $h_{12}, h_{02} \in \mathcal{L}_1$ lift two edges of

the low face – that is, we have an intermediate lifting problem of type

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "h_{12}\cup h_{02}"] \arrow[d, hook] & \mathcal{L} \arrow[d, "\pi"] \\
\Delta[2] \arrow[r, "h|_{012}"'] \arrow[ur, dotted, "H_{012}"] & \mathcal{M}
\end{tikzcd}$$

with solution H_{012}. (This is the first case where we see that it is not enough for π to merely be an inner fibration.) This yields a horn

$$H_{012} \cup h_{023} \cup h_{123} \colon \Lambda^3_2 \to \mathcal{N},$$

which admits a filler $H \in \mathcal{N}$. We observe that the restriction to $\{0\} \times \Delta[2]$ of $H \circ \mathcal{C}_3 \colon \Delta[1] \times \Delta[2] \to \mathcal{N}$, which is precisely H_{012}, factors through $\mathcal{L}$ by construction. Indeed, $(H, 3)$ fills h: $\mathcal{S}_3 \circ \partial_3 \colon [2] \to [1] \times [2]$ sends $i \mapsto (0, i)$, so $d_3 H = H_{012}$ is low:

$$d_3(H, 3) = \pi(H_{012}) = h|_{012} \in \mathcal{M}_2,$$

as desired. In contrast, $\mathcal{S}_3 \circ \partial_i$ for any $i < 3$ clearly hits both $(0, -)$ and $(1, -)$ and so $d_{0/1/2}H$ are vertical. Using $\flat_{3,i} = 2$ for $i < 3$, we see

$$d_i(H, 3) = (h_{0\ldots\widehat{i}\ldots 3}, 2) = h|_{0\ldots\widehat{i}\ldots 3},$$

as desired.

(2) h has an upper face. Analogously, we can assume that the upper face is $h|_{123} \in \mathcal{N}$ whence we have that the sole low vertex is $h|_0$ and that all other faces are vertical with exit index necessarily 1. Exactly like in Case (4) above, we have a horn

$$\widehat{h} \colon \Lambda^3_2 \to \mathcal{E}\mathcal{X}$$

with $\widetilde{h}|_0$ in $\iota(\mathcal{L})$, and with filler, say, $H \in \mathcal{N}_3$. We observe that $(H, 1)$ again fills h. The check is exactly as in said Case.

(3) all faces of h are vertical. Analogously to Case (5) above,

$$h|_{012},$$

necessarily has exit index 2, and

$$h|_{123},\ h|_{023}$$

have exit index 1. The missing face $h|_{013}$ would have had to have index 2 by the same argument. We may thus choose a filler $H \in \mathcal{N}_3$ of $\widetilde{h} \colon \Lambda^3_2 \to \mathcal{N}$ and analogously observe that $(H, 2) \in \mathcal{P}^{\Delta}_2$ fills h.

<u>Horns of arbitrary dimension.</u> Let

$$h \colon \Lambda^n_i \to \mathcal{E}\mathcal{X}$$

be given, with $0 < i < n$. We will adopt the notation and results from the cases of inner 2- and 3-horns treated above. Suppose

(1) h has a low face, which is necessarily $h|_{0\ldots n-1} \in \mathcal{M}_{n-1}$; w.l.o.g., $h|_n \in \mathcal{N}_0$ is the sole upper vertex, and all other faces are vertical. Let us

write $h|_{\widehat{j}}$ for

$$h|_{0\ldots\widehat{j}\ldots n} = h \circ \partial_j \colon \Delta[n-1] \to \mathcal{EX}$$

when that makes sense ($j \neq i$). We have that each vertical face

$$h|_{\widehat{k}} \in \mathcal{P}^{\Delta}_{n-2} \subset \mathcal{EX}_{n-1},\ i \neq k < n$$

has the $(n-2)$-face $h|_{\widehat{k}\widehat{n}}$ in common with the low $h|_{\widehat{n}}$, which therefore gives a lift

$$h_{\widehat{k}\widehat{n}} \in \mathcal{L}_{n-2}$$

to $\mathcal{L}$ thereof, where we wrote

$$h|_{\widehat{k}} = (h_{\widehat{k}}, e)$$

and $(h_{\widehat{k}})_{\widehat{n}} = \iota(h_{\widehat{k}\widehat{n}})$. As each $h|_{\widehat{k}}$ itself has a low face, its exit index is necessarily maximal, i.e.,

$$e = n-1.$$

Now, we obtain the intermediate lifting problem

$$\begin{tikzcd} \Lambda^{n-1}_i \arrow[r, "\bigcup_{i\neq k\in[n-1]} h|_{\widehat{k}\widehat{n}}"] \arrow[d, hook] & \mathcal{L} \arrow[d, "\pi"] \\ \Delta[n-1] \arrow[r, "h|_{\widehat{n}}"'] \arrow[ur, dotted, "H_{\widehat{n}}"] & \mathcal{M} \end{tikzcd} \tag{3.2.12}$$

with solution $H_{\widehat{n}}$. (It is imperative here that π be a right fibration and not just an inner one, since $i = n-1$ is allowed.) This yields the horn

$$\iota(H_{\widehat{n}}) \cup \bigcup_{i\neq k\in[n-1]} \iota(h|_{\widehat{k}\widehat{n}}) \colon \Lambda^n_i \to \mathcal{N}$$

which has a filler $H \in \mathcal{N}_n$.

In fact, (H, n) fills h: the restriction of

$$H \circ \mathcal{C}_n \colon \Delta[1] \times \Delta[n-1] \to \mathcal{N}$$

to $\{0\} \times \Delta[2]$ is $\iota(H_{\widehat{n}})$, which factors through $\mathcal{L}$ by construction. Further,

$$\mathcal{S}_n \circ \partial_n \colon [n-1] \to [1] \times [n-1]$$

sends $n > j \mapsto (0, j)$, so $d_n H$ is low, whence

$$d_n(H, n) = \pi(H_{\widehat{n}}) = h|_{\widehat{n}},$$

as desired. Further, when $k < n$, we have $\mathcal{S}_n \circ \partial_j$ hitting both $\{0\} \times [n-1]$ and $\{1\} \times [n-1]$, so each $d_k H$ is vertical. Using $\flat_{n,k} = n-1$ for $k < n$, we have

$$d_k(H, n) = (h_{\widehat{k}}, n-1) = h|_{\widehat{k}},$$

also as desired.

(2) h has an upper face, which is necessarily $h|_{\widehat{0}} \in \mathcal{N}_{n-1}$; w.l.o.g., $h|_0 \in \mathcal{M}_0$ is the sole low vertex, and all other faces $h|_{\widehat{k}} = (h_{\widehat{k}}, 1) \in \mathcal{P}^{\Delta}_{n-2}$ are vertical with exit index necessarily minimal.

Now, h is given by a horn $\widetilde{h}\colon \Lambda^n_i \to \mathcal{N}$ with $\widetilde{h}|_0 \in \iota(\mathcal{L}_0)$. Taking a filler H of $\widetilde{h}$, we see that $(H,1)$ fills h: the restriction of $\mathcal{C}_1\colon [1]\times[n-1] \to [n]$ to $\{0\}\times[n-1]$ hits only 0, so $H\circ\mathcal{C}_1$ factors through the mapping cocylinder $\mathcal{P}$ by construction, independently of the choice of filler H. Further, $\mathcal{S}_1\colon [n]\to[1]\times[n-1]$ sends only 0 to $\{0\}\times[n-1]$ while $\mathcal{S}_1\circ\partial_0$ factors through $\{1\}\times[n-1]$. This means d_0H is upper, so

$$d_0(H,1) = h|_{\widehat{0}},$$

as desired, and finally

$$d_k(H,1) = (d_kH, \flat_{1,k}) = (h_{\widehat{k}}, 1) = h|_{\widehat{k}}$$

for every $k \geq 1$, also as desired.

(3) all faces of h are vertical. Then $h|_0 \in \mathcal{M}_0$ is low and $h|_n \in \mathcal{N}_0$ upper, and moreover there must exist an index $1 \leq e \leq n$ such that

$$h|_j \in \mathcal{M}_0 \text{ for } j < e \text{ and } h|_j \in \mathcal{N}_0 \text{ for } j \geq e$$

(we had $e = 2$ for 3-horns of both varieties discussed above) for otherwise there would exist a pair $0 < j < j' < n$ such that $h|_j \in \mathcal{N}_0$ while $h|_{j'} \in \mathcal{M}_0$, which is absurd since the edge $h|_{jj'}$ would be of type $\mathcal{N}\to\mathcal{M}$. Moreover, $e = 1$ resp. $e = n$ are impossible, since then $h|_{\widehat{0}}$ resp. $h|_{\widehat{n}}$ would be low resp. upper. We have obtained

$$1 < e < n.$$

(There is no 2-horn both of whose faces are vertical, so we may assume $n \geq 3$.) Now, we claim that the exit indices of the faces $h|_{\widehat{j}} \in \mathcal{P}^{\Delta}_{n-2}$, $j \neq i$, are determined by this e:

$$h|_{\widehat{j}} = \begin{cases} (h_{\widehat{j}}, e), & j \geq e, \\ (h_{\widehat{j}}, e-1), & j < e. \end{cases} \tag{3.2.13}$$

Indeed, that

$$\mathcal{C}^{n-1}_{\ell}(\{0\}\times[n-2]) = \{0,\dots,\ell-1\}$$

for any $1 \leq \ell \leq n-1$ implies that if $j \geq e$, then $h_{\widehat{j}}\circ\mathcal{C}^{n-1}_e$ factors through $\mathcal{P}$, as does $h|_{\widehat{j}}\circ\mathcal{C}^{n-1}_{e-1}$ if $j < e$. Conversely, suppose $h|_{\widehat{j}}$ has index e': $(h|_{\widehat{j}})|_{0,\dots,e-1}$ must be low, which implies, by the definition of $\mathcal{S}_{e'}$, that $e' \geq e$, and since there are no further low vertices, we have $e' \leq e$.

Now, as h induces (or is rather underlied by) a horn $\widetilde{h}\colon \Lambda^n_i \to \mathcal{N}$, we may choose a filler $H \in \mathcal{N}_3$ and claim that (H,e) in turn is a filler for h. In order to ensure that $H\circ\mathcal{C}_e\colon \Delta[1]\times\Delta[n-1]\to\mathcal{N}$ factors through $\mathcal{P}$, it suffices to observe that the missing face $h|_{\widehat{i}}$ cannot be low, for then the choice of filler H does not affect the factorisation property (in that $\widetilde{h}$ needs filling only away from $\iota(\mathcal{L})$). Indeed, the only such case would be when $i = n$, but h is inner.

Finally, we check the exit indices of the faces of H: since $1 < e < n$, no face of H is low or upper, and (3.2.6) implies $d_j(H, e) = h|_{\widehat{j}}$ due to (3.2.13), as desired. □

Definition 3.2.14. We call a span $\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}$ of ∞-groupoids resp. ∞-categories, with π a Kan resp. right fibration, and ι a cofibration, a *linked ∞-groupoid* or *linked space* resp. *linked ∞-category*, of *depth* 1. We call $\mathcal{EX}$ its *exit path ∞-category*.

We will obtain compatibility with [6, Lemma 3.3.5] in Section 3.3.[10] Since the definition of the exit path ∞-category in [6] coincides, by [6, Lemma 3.3.9] with the Lurie–MacPherson model of [50, App. A] up to equivalence, this will lift to a statement about the Lurie–MacPherson model as well. Let us first briefly discuss how some classical examples fit into this setting. They will be of central importance.

Example 3.2.15 (Bordisms)**.** Since we only explicitly treated depth 1, we restrict ourselves to manifolds with boundary, but the higher-depth treatment of corners is analogous. The linked space corresponding to a (smooth) manifold with boundary $(M, \partial M)$ has lower stratum ∂M, higher stratum $M^\circ = M \backslash \partial M$, link $L = \partial M$, $\pi = \mathrm{id}_{\partial M}$, and $\iota \colon \partial M \hookrightarrow M^\circ$ given by the flow along a nowhere-vanishing inward-pointing vector field along the boundary for a chosen nonzero time. An equivalent way to pick ι is to consider a tubular neighbourhood of the boundary diffeomorphic (via such a vector field) to $\partial M \times [0, 1) \hookrightarrow M$, whose restriction $\partial M \times (0, 1) \hookrightarrow M^\circ$ to positive time hits the interior, and take ι to be the restriction to $\partial M \times \{\frac{1}{2}\}$.

Example 3.2.16 (Defects)**.** With a smooth submanifold $\Sigma \subset M$ of positive codimension we may associate a nontrivial linked space with lower stratum Σ and higher stratum $M \setminus \Sigma$. The link is given by $\mathbb{SN}(\Sigma)$, the sphere bundle of the normal bundle of Σ, with the obvious maps π and ι. For instance, the link of $\mathbf{R} \subset \mathbf{R}^3$ is the open (infinite) cylinder $S^1 \times \mathbf{R}$, whereas the link of $S^1 \subset \mathbf{R}^3$ is a torus.

Example 3.2.17 (Depth-1 stratified Grassmannians)**.** For $n, k \in \mathbf{N}$, consider the span

$$\begin{tikzcd} & B\mathrm{O}(n) \times B\mathrm{O}(k) \arrow[dl, two heads, "\pi"'] \arrow[dr, hook, "\oplus"] & \\ B\mathrm{O}(n) & & B\mathrm{O}(n+k) \end{tikzcd}$$

where π is the coordinate projection and $\oplus$ is induced by direct-summing of vector spaces and the choice of a pairing function (bijection) $\mathbf{N} \times \mathbf{N} \cong \mathbf{N}$. This gives sub-$\infty$-categories of the quasi-category model of the stratified Grassmannian of [7] given in Chapter 4.[11] A higher-depth treatment ought to reconstruct the full ∞-category, but we leave this to future work.

[10] It is clear from the construction that $\mathcal{EX}^{\simeq} \simeq \mathcal{M} \amalg \mathcal{N}$ is the maximal sub-∞-groupoid.

[11] We slightly deviated from the map $\oplus$ used in Chapter 4 by using a pairing function, but only up to an equivalence induced by it.

Any Kan/right fibration π alone, or any cofibration ι alone gives an example with a trivial choice for the other leg: the identity cofibration or the final fibration to the point:

$$\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{=} \mathcal{L}$$

or

$$* \leftarrow \mathcal{L} \xrightarrow{\iota} \mathcal{N}.$$

Any ∞-category $\mathcal{X}$ gives a linked ∞-category

$$\emptyset \leftarrow \emptyset \rightarrow \mathcal{X}$$

with $\mathcal{EX}(\emptyset \leftarrow \emptyset \rightarrow \mathcal{X}) \simeq \mathcal{X}$. The other trivial construction

$$* \leftarrow \mathcal{X} \xrightarrow{=} \mathcal{X}$$

corresponds to taking the open cone of $\mathcal{X}$ – literally in the ordinary stratified setting for $\mathcal{X} = \mathrm{Exit}(X)$, recalling that $\mathrm{Exit}(C(X)) \simeq \mathrm{Exit}(X)^{\triangleleft}$ – in that

$$* \in \mathcal{EX}\left(* \leftarrow \mathcal{X} \xrightarrow{=} \mathcal{X}\right) \simeq \mathcal{X}^{\triangleleft}$$

is initial. We prove this in Corollary 3.3.3.

3.3. Linked morphism spaces

Notation. Given an embedding $\iota\colon \Sigma \hookrightarrow N$ and a point $q \in N$, we let $P(N)_{\Sigma,q} = P_{\Sigma,q}$ denote the space of paths in N that start in $\iota(\Sigma)$ and end in the point q, equipped with the compact-open topology. We use analogous notation when we work with a cofibration ι of simplicial sets.

The following result formalises and confirms the intuition that the link represents an infinitesimal expansion of the lower stratum into the higher stratum. More precisely, it is a pointwise version of that sentiment. The proof is contained, in essence, in the proof of Theorem 3.2.11, but we will extract it to make an indepedent reading possible.

Theorem 3.3.1. *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ be a linked ∞-category, and $p \in \mathcal{M}$ and $q \in \mathcal{N}$ points in the two strata. We then have an equivalence*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \mathcal{P}_{\mathcal{L}_p,q}$$

between the morphism space in $\mathcal{EX}$ from p to q and that of paths in $\mathcal{N}$ that start in the embedded fibre $\iota(\mathcal{L}_p)$, where $\mathcal{L}_p = \{p\} \times_{\mathcal{M}} \mathcal{L}$, and end in q.

Proof. We will work with a model for morphism spaces that makes the proof particularly simple: by [52, 01L5], the morphism space in $\mathcal{EX}$ is equivalent to the *right-pinched* morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{EX}}(p, q) := \{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$. We will observe directly that $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is in fact isomorphic to $\mathcal{P}_{\mathcal{L}_p,q} = \mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)$. Indeed, at vertex level, the bijection

$$\left(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)\right)_0 \cong \left(\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)\right)_0$$

is clear: recalling that non-invertible 1-paths in $\mathcal{EX}$ are elements of $\mathcal{P}_0^{\Delta} \subset \mathcal{N}_1$ (as the exit index is necessarily 1 in this degree), let $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$. For

$p = d_1^{\mathcal{EX}}(\gamma, 1) \overset{\text{def}}{=} \pi(d_1^{\mathcal{N}}(\gamma))$ to hold, we must have $d_1^{\mathcal{N}}(\gamma) \in \iota(\mathcal{L}_p)$. Similarly, $d_0^{\mathcal{EX}}(\gamma, 1) \overset{\text{def}}{=} \iota(d_0^{\mathcal{N}}(\gamma))$, which yields the bijection.

Let now $k > 0$ and consider an exit $(k+1)$-path $(\gamma\colon \Delta[k] \star \Delta[0] \to \mathcal{N}, j)$ in $(\mathcal{EX}/q)_k \subset \mathcal{EX}_{k+1}$. Asking that (γ, j) be in $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is equivalent to asking that

(1) its $\mathcal{N}$-face
$$\Delta[k] \hookrightarrow \Delta[k] \star \Delta[0] \xrightarrow{\gamma} \mathcal{N},$$
which is $d_{k+1}^{\mathcal{N}}(\gamma)$ under the standard identification $\Delta[k] \star \Delta[0] \simeq \Delta[k+1]$, is bottom, as by construction only then can the corresponding $\mathcal{EX}$-face be in $\mathcal{M}_k \subset \mathcal{EX}_k$;
(2) and that it lies in particular in $\iota(\mathcal{L}_p)$.

Condition (1) implies moreover that $d_\ell^{\mathcal{N}}(\gamma) \in \mathcal{N}_k$ is vertical for all $\ell < k+1$, since all other faces include the tip $\Delta[0] \hookrightarrow \Delta[k] \star \Delta[0]$ given by q, whence they are necessarily not bottom; and if some $d_\ell(\gamma)$ was top, that would contradict the bottomness of its (unique) common $(k-1)$-face with $d_{k+1}^{\mathcal{N}}(\gamma)$. In fact, (γ, j) has no n-face that is top once $n > 0$: given $\Delta[n] \hookrightarrow \Delta[k+1]$, there is necessarily a vertex in $d_{d+1}^{\mathcal{N}}(\gamma)$ that is hit by it.

But then the exit index j must be maximal: $j = k+1$. For if not, then there would exist at least one top n-face for $n > 0$, the largest such, with $n = k+1-j$, for instance, being specified by $[n] \hookrightarrow [k+1]$, $\alpha \mapsto \ell + \alpha$. We thus obtain a bijection $(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_k \cong (\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q))_k$ in a fashion similar to the bijection of vertices: we have reduced exit paths (γ, j) in question on the LHS to those of index $k+1$, and so to only a subset of $\mathcal{N}_{k+1}$, and specifically those such that $d_{k+1}^{\mathcal{N}}(\gamma) \in \mathcal{L}_p$. These are exactly the elements of the RHS. Finally, it is a direct check that $(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_* \to (\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q))_*$ is functorial; for instance, any vertical face of such a $(\gamma, k+1)$ is again of maximal index: using the formulas in the proof of Lemma 3.2.3, we have $\flat_{k+1,i}^{k+1} = k$ and, and as for degeneracies, $\sharp_{k+1,i}^{k+1} = k+2$, for all $i < k+1$. □

Corollary 3.3.2. *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ be a linked space with $\mathcal{M}$ and $\mathcal{N}$ connected, and $p \in \mathcal{M}$, $q \in \mathcal{N}$. Then, π is an equivalence if and only if*
$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}.$$
Here, $\Omega\mathcal{N}$ denotes the based loop space of $\mathcal{N}$.

Proof. The fibre at any point of the source evaluation $\mathcal{P}_{\mathcal{L}_p, q} \to \mathcal{L}_p$ is equivalent to $\Omega\mathcal{N}$. Thus, the homotopy long exact sequence of this fibration implies that the fibre inclusion induces isomorphisms $\pi_*(\Omega\mathcal{N}) \cong \pi_*(\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q))$ iff $\pi_*(\mathcal{L}_p) \cong *$. Thus, $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}$ iff π is a trivial Kan fibration. But, by [52, 00X2], π is a trivial Kan fibration iff it is an equivalence. □

We interpret this as saying that the spaces of non-invertible paths in a linked space is are at their largest when π is an equivalence: there are just as many as there are paths in the higher stratum. This is the case, for instance, when $\mathfrak{S}$ is induced by a manifold with boundary, as in Example 3.2.15. We have a maximal simplification in the other extreme, namely when π is trivial.

Corollary 3.3.3. *For a linked space of type*

$$* \leftarrow \mathcal{N} \xrightarrow{\mathrm{id}} \mathcal{N}$$

we have

$$\mathrm{Hom}_{\mathcal{EX}}(*, q) \simeq *.$$

Consequently, when $\mathcal{N} = \mathrm{Sing}_\bullet(N)$ *for* N *a smooth manifold, we have*

$$\mathrm{Exit}(C(N)) \simeq \mathcal{EX},$$

where the left-hand side is the exit path ∞*-category of the conically-smooth open cone*

$$C(N) = * \amalg_{\{0\}\times N} ([0,1) \times N)$$

on N *with its canonical stratification over* $\{0 < 1\}$*.*

Proof. We have $\mathcal{L}_* = \mathcal{N}_* = \mathcal{N}$ and so $\mathcal{P}_{\mathcal{L}_*,q} \simeq \mathcal{N}/q \simeq *$ since $\mathcal{N}$ is an ∞-groupoid ([52, 018Y]). This implies $\mathcal{EX} \simeq \mathrm{Sing}_\bullet(N)^{\triangleleft}$. The latter agrees with the LHS by [6, Proposition 3.3.8].[12] ☐

Since the link of the cone locus $*$ and the interior of $C(N)$ is N itself, one could consider $* \leftarrow N \hookrightarrow N \times \mathbf{R}$ to be the natural linked space model for the open cone. Mutatis mutandis, the proof of Corollary 3.3.3 implies that this modification changes the exit path ∞-category only up to equivalence. It remains desirable, then, to reach a more systematic understanding of the linked incarnation of $\mathbf{R}^1$-invariance in the conically-smooth theory.

3.4. The space of paths between strata

In this section, let $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked space. We will generalise Theorem 3.3.1 in this setting and identify L, up to equivalence, with the space of paths in $\mathcal{EX}$ that start in M and end in N.

Let $p \in M$ and $q \in N$. We have $(L_p \downarrow q)^N = P_{L_p,q} \simeq \mathrm{Hom}(p,q) = (p \downarrow q)^{\mathcal{EX}}$. Formally, varying p should give an equivalence

$$(L \downarrow q)^N = P_{L,q} \simeq (L \downarrow q)^{\mathcal{EX}} \simeq (M \downarrow q)^{\mathcal{EX}}$$

and then varying q should give

$$L \simeq (L \downarrow N)^N = P_L \simeq (M \downarrow N)^{\mathcal{EX}}.$$

Theorem 3.4.1. $L \simeq (M \downarrow N)$*.*

Our first lemma shows that Remark 3.1.2 holds with ∞-groupoids just as it does with topological spaces. We include a proof for completeness.

Lemma 3.4.2. *Let* P_ι *be the mapping cocyclinder as in Definition 3.1.1. If* L *and* N *are* ∞*-groupoids, then* $P_\iota \simeq L$*.*

[12] More precisely, this is an equivalence of quasi-categories for Lurie's model from [50], or, after translating to the complete Segal space model and using [6, Lemma 3.3.9], with that of Ayala et al.

PROOF. Since N is an ∞-groupoid, we observe that the source map $N^{\Delta[1]} \to N^{\{0\}}$ is a Kan fibration between Kan complexes. Moreover, each fibre $N_p^{\Delta[1]} \simeq p/N$, $p \in N_0$, is contractible by virtue of being an under-∞-groupoid ([52, 018Y]). This verifies condition (4) of [52, 00X2], which implies that $N^{\Delta[1]} \to N^{\{0\}}$ is an equivalence, or equivalently (by the same cited result), a trivial Kan fibration. Kan fibrations are stable under pullback [52, 00T5], so the natural map $\mathbf{s}\colon P_\iota \to L$ is a Kan fibration. Finally, as trivial Kan fibrations pull back to trivial Kan fibrations, $\mathbf{s}$ is one such. As it is in particular a Kan fibration, [52, 00X2] implies that $\mathbf{s}$ is an equivalence. □

As its proof shows, the preceding lemma is a generalisation of the fact that under-∞-groupoids are contractible, which is the special case when L is a point. Since under-∞-categories can be far from contractible, there is no reason to expect that Theorem 3.4.1 holds for linked ∞-categories. Indeed, most linked ∞-categories where for instance $\mathcal{N}$ contains a non-invertible morphism from $\iota(\mathcal{L})$ to $\mathcal{N}$ provide counterexamples, e.g., $(\{0\} \leftarrow \{0\} \hookrightarrow \Delta[2])$. However, the following weaker result holds for any linked ∞-category.

Lemma 3.4.3. *$(L \downarrow N) \simeq (M \downarrow N)$.*

PROOF. We will in fact give an isomorphism. Observe $(M \downarrow N)_0 = \{\alpha \in \mathcal{EX}_1 : d_1\alpha \in M,\ d_0\alpha \in N\} = \mathcal{P}_0^\Delta$ so that $\alpha \in (M \downarrow N)_0$ iff $\alpha = (\Gamma, 1)$ with $\Gamma \in (P_L)_0 = (L \downarrow N)_0$. Thus, we have the map $(M \downarrow N)_0 \to (L \downarrow N)_0$, $(\Gamma, 1) \mapsto \Gamma$. This gives a bijection $(M \downarrow N)_0 \cong (L \downarrow N)_0$. A similar construction works in any dimension.

Indeed, let $\alpha\colon \Delta[1] \times \Delta[n] \to \mathcal{EX}$ be an element of $(M \downarrow N)_n$, i.e., $\mathrm{ev}_0\alpha \in M_n$, $\mathrm{ev}_1\alpha \in N_n$. Then its restriction along any exit shuffle $\mathcal{S}_j\colon \Delta[n+1] \hookrightarrow \Delta[1] \times \Delta[n]$ where $j \in \{1, \dots, n+1\}$ is vertical, since shuffles hit both ends of the cylinder. Thus,

$$\alpha|_{\mathcal{S}_j} = (\alpha_j, j) \in \mathcal{P}_n^\Delta \quad \text{with} \quad \alpha_j|_{0,\dots,j-1} \in \iota(L)_{j-1}.$$

We will observe that the collection $\{\alpha_j\} \subset N_{j+1}$ assembles to give a map

$$A\colon \Delta[1] \times \Delta[n] \to N$$

which descends to $(L \downarrow N)_n$. Indeed, setting

$$A|_{\mathcal{S}_j} \coloneqq \alpha_j$$

defines A on every non-degenerate $(n+1)$-simplex of $\Delta[1] \times \Delta[n]$ consistently since α itself is well-defined. More precisely, let $\Theta\colon \Delta[\theta] \hookrightarrow \Delta[n+1]$ be some common simplicial subset of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ in $\Delta[1] \times \Delta[n]$. We must show that $\Theta^* A|_{\mathcal{S}_j} = \Theta^* A|_{\mathcal{S}_{j'}}$, but α already satisfies this, i.e., $\Theta^*\alpha|_{\mathcal{S}_j} = \Theta^*\alpha|_{\mathcal{S}_{j'}}$, so that in particular $\Theta^*\alpha_j = \Theta^*\alpha_{j'}$ in $N_{n+1-(j'-j)}$. This yields $A \in N_n^{\Delta[1]} = (N \downarrow N)_n$. Since $\mathrm{ev}_0 A|_{\mathcal{S}_j} = \alpha_j|_{0,\dots,j-1} \in \iota(L)_{j-1}$ for any exit index j as remarked above, we have $\mathrm{ev}_0 A \in \iota(L)_n$, giving $A \in (L \downarrow N)_n$. We have thus constructed a map

$$\Phi\colon (M \downarrow N) \to (L \downarrow N),$$
$$\alpha \mapsto \Phi(\alpha) \coloneqq A.$$

As for the inverse, let $\beta\colon \Delta[1]\times\Delta[n]\to N$ be an element of $(L\downarrow N)_n$, i.e., $\mathrm{ev}_0\beta\in\iota(L)_n$, and let $j<j'$ be exit indices as above. Set

$$B|_{\mathcal{S}_j} \coloneqq (\beta|_{\mathcal{S}_j}, j)\in\mathcal{P}^{\Delta}_n.$$

This is well-defined, since

$$\mathrm{ev}_0\mathcal{C}_j\beta|_{\mathcal{S}_j} = \beta|_{\mathcal{S}_j}|_{0,\ldots,j-1} = (\mathrm{ev}_0\beta)|_{0,\ldots,j-1}\in\iota(L)_{j-1},$$

so that $(\beta|_{\mathcal{S}_j}, j)$ is indeed an exit path of index j. We have $\Theta^*B|_{\mathcal{S}_j} = \Theta^*B|_{\mathcal{S}_{j'}}$ since $\Theta^*\beta|_{\mathcal{S}_j} = \Theta^*\beta|_{\mathcal{S}_{j'}}$. Thus, $\{B|_{\mathcal{S}_j}\}$ assembles to give a map $B\colon\Delta[1]\times\Delta[n]\to\mathcal{EX}$. Moreover, B descends to $(M\downarrow N)_n$, since the initial vertex of every $(\beta|_{\mathcal{S}_j}, j)$ – or of any exit path, for that matter – is low, so that all vertices of ev_0B are low. We have thus constructed a map

$$\begin{aligned}\Psi\colon (L\downarrow N) &\to (M\downarrow N),\\ \beta &\mapsto \Psi(\beta)\coloneqq B.\end{aligned}$$

We may check directly that Φ and Ψ are mutual inverses. Indeed, in any dimension, we have $\Psi\Phi(\alpha)|_{\mathcal{S}_j} = (\Phi(\alpha)|_{\mathcal{S}_j}, j) = (\alpha_j, j) = \alpha|_{\mathcal{S}_j}$, so $\Psi\Phi=\mathrm{id}$. Conversely, $\Phi\Psi(\beta)|_{\mathcal{S}_j} = \Psi(\beta)_j = \beta|_{\mathcal{S}_j}$, so $\Phi\Psi=\mathrm{id}$. $\square$

Proof of Theorem 3.4.1. We have $P_\iota = L\times_{N^{\{0\}}} N^{\Delta[1]}\cong(L\downarrow N)$. The statement follows by composing Lemma 3.4.2 and Lemma 3.4.3. $\square$

3.5. A point and a line in space

In this slightly less formal section we wish to see how well Theorem 3.4.1 fares in the context of an iterated application of $\mathcal{EX}$. It is not needed in the rest of this work and can thus be skipped.

We will consider the example of $\mathbf{R}^3$, stratified, in the ordinary sense, as $\{0\}\subset\mathbf{R}\subset\mathbf{R}^3$ where the middle term is a line through the origin. The strata of this space are $\{0\}$, $\mathbf{R}^\times\coloneqq\mathbf{R}\smallsetminus\{0\}$, and $\mathbf{R}^3\smallsetminus\mathbf{R}$. The stratifying map $\mathbf{R}^3\to\mathcal{P}$ is the obvious one to the poset $\mathcal{P}=\{0\prec 1\prec 3\}$.

Taking $\mathbf{R}_*$ to consist of $\{0\}$ and $\mathbf{R}^\times$, i.e., $\mathbf{R}$ stratified as $\{0\}\subset\mathbf{R}$, we have that the link between $\{0\}$ and $\mathbf{R}^\times$ consists of two points. If we were to disregard the stratification on $\mathbf{R}$, the link between it and $\mathbf{R}^3$ would be $S^1\times\mathbf{R}$. To verify these links, see Example 3.2.16.

We may now construct $\mathcal{EX}(\mathbf{R}_*)$. However, the 'link projection' $S^1\times\mathbf{R}\twoheadrightarrow\mathbf{R}$ does not descend to an ∞-functor $S^1\times\mathbf{R}\to\mathcal{EX}(M')$ since the target is not an ∞-groupoid, but there is an induced stratification[13] on $S^1\times\mathbf{R}$ through the underlying map $S^1\times\mathbf{R}\to\mathbf{R}$ given by composing with the latter's stratification:

$$S^1\times\mathbf{R}\to\mathbf{R}\to\mathcal{P}|_{\mathbf{R}}=\{0\prec 1\}.$$

Let us denote the resulting linked space (using Example 3.2.16 again) by $\mathcal{L}_{\mathbf{R}_*}$. Now, $S^1\times\mathbf{R}\twoheadrightarrow\mathbf{R}$ does descend to an ∞-functor

$$\mathcal{EX}(\mathcal{L}_{\mathbf{R}_*})\twoheadrightarrow\mathcal{EX}(\mathbf{R}_*).$$

[13]realising a weakly-constructible bundle

which is a right fibration by virtue of being a Kan fibration. Moreover, the embedding $S^1 \times \mathbf{R} \hookrightarrow \mathbf{R}^3 \smallsetminus \mathbf{R}$ naturally descends to an ∞-functor $\mathcal{EX}(\mathcal{L}_{\mathbf{R}_*}) \hookrightarrow \mathbf{R}^3 \smallsetminus \mathbf{R}$, which is trivially a cofibration, so that we can set up the linked ∞-category

$$\begin{tikzcd} & \mathcal{EX}(\mathcal{L}_{\mathbf{R}_*}) \arrow[dl, two heads] \arrow[dr, hook] & \\ \mathcal{EX}(\mathbf{R}_*) & & \mathbf{R}^3 \smallsetminus \mathbf{R} \end{tikzcd}$$

and denote it by, say, $\mathbf{3}_{(\mathbf{1_0})}$. This is the result of 'parenthesising' the chain inclusions as

$$(\{0\} \subset \mathbf{R}) \subset \mathbf{R}^3$$

and applying $\mathcal{EX}$ accordingly.

Alternatively, we may parenthesise as

$$\{0\} \subset (\mathbf{R} \subset \mathbf{R}^3).$$

That is, ignore the stratification on $\mathbf{R}_*$ to begin with and construct $\mathcal{EX}(\mathbf{R}^3_{\mathbf{R}})$ first, where $\mathbf{R}^3_{\mathbf{R}}$ is (using Example 3.2.16 again) the linked space

$$\mathbf{R} \twoheadleftarrow S^1 \times \mathbf{R} \hookrightarrow \mathbf{R}^3 \smallsetminus \mathbf{R}.$$

Now, the link of $\{0\} \subset \mathbf{R}^3$ is S^2. We see that this gives rise to the opposite problem to the one in the paragraph above: the projection $S^2 \to \{0\}$ is fine, but the embedding $S^2 \hookrightarrow \mathbf{R}^3$ does not descend to an ∞-functor $S^2 \hookrightarrow \mathcal{EX}(\mathbf{R}^3_{\mathbf{R}})$. Still, the same solution is available: we may compose with the stratifying map defining $\mathbf{R}^3_{\mathbf{R}}$ to obtain an induced stratification

$$S^2 \hookrightarrow \mathbf{R}^3 \to \mathcal{P}' = \{1 \prec 3\},$$

where $\mathcal{P}'$ stratifies $\mathbf{R}^3$ in the obvious way by mapping the chosen line to 1 and its complement to 3. The preimage of 1 in S^2 consists of the two intersection points, say p and q. Thus, S^2 is broken into two strata, $\{p,q\}$ and $S^2 \smallsetminus \{p,q\}$. Let us denote the resulting linked space (using Example 3.2.16 yet again) by

$$\mathcal{S}^2_{p,q} = \left(\{p,q\} \twoheadleftarrow S^1_p \amalg S^1_q \hookrightarrow S^2 \smallsetminus \{p,q\}\right).$$

The two circles sit within coordinate neighbourhoods around the two points.

There is a span map

$$\mathcal{S}^2_{p,q} \hookrightarrow \mathbf{R}^3_{\mathbf{R}}$$

whose link component can be given by shrinking the circle factor of the target's link $S^1 \times \mathbf{R}$ – obtaining $s^1 \times \mathbf{R}$ – such that it intersects S^2 in exactly two circles around p and q, and redefining S^1_p and S^1_q to be these circles. Then, $S^1_p \amalg S^1_q \hookrightarrow s^1 \times \mathbf{R}$ can be chosen to be simply the identity inclusion. The map $\{p,q\} \hookrightarrow \mathbf{R}$, however, cannot be the identity, since then the side

$$\begin{tikzcd} S^1_p \amalg S^1_q \arrow[r, hook] \arrow[d, two heads] & s^1 \times \mathbf{R} \arrow[d, two heads] \\ \{p,q\} \arrow[r, hook, dashed] & \mathbf{R} \end{tikzcd}$$

of the span-map-to-be would not commute. (It commutes only 'in the limit as s^1 approaches a point.') However, there is an embedding $\{p, q\} \hookrightarrow \mathbf{R}$ induced by this diagram after choosing elements in the fibres as depicted. This is easily seen to be well-defined. Furthermore, the other side

$$\begin{tikzcd} S^1_p \amalg S^1_q \arrow[r, hook] \arrow[d, hook] & s^1 \times \mathbf{R} \arrow[d, hook] \\ S^2 \smallsetminus \{p, q\} \arrow[r, hook] & \mathbf{R}^3 \smallsetminus \mathbf{R} \end{tikzcd}$$

commutes since every map involved is given by the identity inclusion. We have thus constructed the desired span map.[14]

This map is term-wise an embedding, and so induces a cofibration

$$\mathcal{EX}(\mathcal{S}^2_{p,q}) \hookrightarrow \mathcal{EX}(\mathbf{R}^3_{\mathbf{R}})$$

of ∞-categories. We can thus set up the linked ∞-category

$$\begin{tikzcd} & \mathcal{EX}(\mathcal{S}^2_{p,q}) \arrow[dl, two heads] \arrow[dr, hook] & \\ \{0\} & & \mathcal{EX}(\mathbf{R}^3_{\mathbf{R}}) \end{tikzcd}$$

and denote it by, say, $(\mathbf{3_{1)_0}}$.

The immediate question is whether there is an equivalence

$$\mathcal{EX}\big((\mathbf{3_{1)_0}}\big) \simeq \mathcal{EX}\big(\mathbf{3_{(1_0)}}\big)$$

of ∞-categories.

There is a bijection between their vertices but only in a useless sense: those of the LHS is given by $\{0\}_0 \amalg (\mathbf{R}_0 \amalg (\mathbf{R}^3 \smallsetminus \mathbf{R})_0)$, and those of the RHS by $(\{0\}_0 \amalg \mathbf{R}_0^\times) \amalg (\mathbf{R}^3 \smallsetminus \mathbf{R})_0$, so the former counts 0 twice. This is a strong indication that the RHS is the correct order of iteration. We invite the reader to check that the RHS has the correct links as well, and in fact there is no such equivalence as mentioned above. We conjecture that this procedure – starting with deepest strata and applying $\mathcal{EX}$ pairwise while keeping track of induced stratifications on higher links, and iterating $\mathcal{EX}$ – recovers the exit path ∞-categories of (conical, conically-smooth or homotopical) stratified spaces in higher depth. But what, regardless, is the meaning of the LHS?

[14]Alternatively, we could have included p and q into $\mathbf{R}$ by the identity but chosen a different embedding for the sphere, one flattened near the two circles.

CHAPTER 4

Quasi-(de)looping

In this chapter, we will take the first step towards transporting the tangential theory of conically-smooth stratified spaces to the linked setting.

4.1. The additive Grassmannian

Let H be a separable real Hilbert space of countably-infinite dimension, so, up to isometric isomorphism, the real sequence space ℓ^2.

Definition 4.1.1. For $k \in \mathbf{N}$, $B\mathrm{O}(k) := \mathrm{Gr}_k(H)$ denotes the Grassmannian of k-dimensional subspaces of H.

$B\mathrm{O}(k)$ is an infinite-dimensional (Hilbert) manifold modelled on H, and, thus topologised, is homotopy-equivalent (e.g. [61, 4 ff.] combined with Whitehead's theorem) to the colimit infinite Grassmannian

$$\mathrm{Gr}_k(\mathbf{R}^\infty) = \operatorname{colim} \mathrm{Gr}_k(\mathbf{R}^n)$$

along the closed embeddings

$$\mathrm{Gr}_k(\mathbf{R}^n) \hookrightarrow \mathrm{Gr}_k(\mathbf{R}^{n+1})$$

given by the first-coordinate inclusions $\mathbf{R}^n \hookrightarrow \mathbf{R}^{n-1}$. For our purposes, $\mathbf{R}^\infty$, H, and ℓ^2 are interchangeable.

Notation 4.1.2. $B\mathrm{O}_{\amalg} := \coprod_{k \geq 0} B\mathrm{O}(k)$, $B\mathrm{O}_{\amalg}^{+} := \coprod_{k \geq 1} B\mathrm{O}(k)$.

Remark 4.1.3. The purpose of the notation is to distinguish it from the (connected component of the zeroeth space of the real K-theory) spectrum $B\mathrm{O}$, which is given by a non-discrete colimit.

The aim of this section is to define a monoidal structure based on direct-summing of vector spaces, in the spirit of the direct-summing maps

$$\oplus\colon \mathrm{Gr}_k(\mathbf{R}^n) \times \mathrm{Gr}_l(\mathbf{R}^m) \to \mathrm{Gr}_{k+l}(\mathbf{R}^{n+m})$$

Passing to infinite Grassmannians, these give maps

$$B\mathrm{O}(k) \times B\mathrm{O}(l) \to \mathrm{Gr}_{k+l}(H \oplus H).$$

Choosing an isomorphism $H \oplus H \cong H$ yields a map

$$B\mathrm{O}(k) \times B\mathrm{O}(l) \to B\mathrm{O}(k+l),$$

which defines a map

$$\oplus\colon B\mathrm{O}_{\amalg} \times B\mathrm{O}_{\amalg} \to B\mathrm{O}_{\amalg} \tag{4.1.4}$$

connected-componentwise.

The problem with this map is that there is no choice of an isomorphism $H \oplus H \cong H$ that would make the map above associative, so it does not promote $B\mathrm{O}_{\mathrm{II}}$ to a topological monoid. For our purposes, it will suffice to point out that an isomorphism $H \oplus H \cong H$, or equivalently a pairing function (bijection) $\mathbf{N} \times \mathbf{N} \cong \mathbf{N}$ cannot be associative, as this would contradict injectivity. In order to attain hands-on access to the stratified Grassmannian, we have chosen to strictify (a.k.a. rigidify) $(B\mathrm{O}_{\mathrm{II}}, \oplus)$ in a certain way instead. This involves a trade-off: it does give a topological monoid, but also introduces some redundancy.

Notation 4.1.5. $B\mathrm{O}^N(k) \coloneqq \mathrm{Gr}_k(H^{\oplus N})$.

Notation 4.1.6. $B\mathrm{O}_{\mathrm{II}}^{\infty} \coloneqq \{0\} \amalg \coprod_{N\geq 1} \coprod_{k \geq 1} B\mathrm{O}^N(k)$.

Remark 4.1.7. Of course, each $B\mathrm{O}^N(k)$ is equivalent (even homeomorphic) to $B\mathrm{O}(k) = B\mathrm{O}^1(k)$, but this is non-canonical. Thus, with some choice of a pairing function and some choice of parenthesisation for large exponents, we have

$$B\mathrm{O}_{\mathrm{II}}^{\infty} \simeq \{0\} \amalg \mathbf{Z}_+ \times B\mathrm{O}_{\mathrm{II}}^+.$$

We separated the zero vector space singleton $\{0\} = B\mathrm{O}(0) = B\mathrm{O}^N(0)$ from the disjoint union so as not to count it separately for each $N \geq 1$.

Construction 4.1.8. Direct-summing of vector spaces gives maps

$$B\mathrm{O}^N(k) \times B\mathrm{O}^M(l) \to B\mathrm{O}^{N+M}(k+l),$$

which define a map

$$\oplus\colon B\mathrm{O}_{\mathrm{II}}^{\infty} \times B\mathrm{O}_{\mathrm{II}}^{\infty} \to B\mathrm{O}_{\mathrm{II}}^{\infty}$$

connected-componentwise. The zero vector space acts as the identity. This is easily seen to be associative.

Remark 4.1.9. The canonical associativity of direct-summing of vector bundles on (paracompact Hausdorff) spaces translates to a monoidal structure on $B\mathrm{O}_{\mathrm{II}}$ (or its stable version $B\mathrm{O}$) only up to coherent homotopy. A systematic treatment in this direction, i.e., the theory of $\mathbb{E}_\infty$-rings and its application to spectra, is laid out in [50]; see also [68]. The $\mathbb{E}_\infty$-structure on $B\mathrm{O}_{\mathrm{II}}$ is parametrised, at arity n, by the (contractible) space of embeddings $H^n \hookrightarrow H$.

Construction 4.1.8 is considered from a slightly different point of view in [68, §2], where $(B\mathrm{O}_{\mathrm{II}}^{\infty}, \oplus)$ with H relaxed to a vector space variable is called the *additive Grassmannian*.

The construction and result of this note apply immediately to the other varieties of Grassmannians such as the oriented, quaternionic, etc.

We will now deloop the topological monoid $(B\mathrm{O}_{\mathrm{II}}^{\infty}, \oplus)$. By N^{hc} we denote the homotopy-coherent nerve, which is recalled in Section 2.2.1.

Definition 4.1.10. By $B^{\oplus}\mathrm{O}$ we denote the Kan-enriched category with a single object $*$, endomorphism space $B\mathrm{O}_{\mathrm{II}}^{\infty}$, and composition $\oplus$.

Definition 4.1.11. $\mathcal{B}^{\oplus}\mathrm{O} \coloneqq \mathrm{N}^{\mathrm{hc}}(B^{\oplus}\mathrm{O})$.

Remark 4.1.12. Note that $\mathcal{B}^{\oplus}\mathrm{O}$ is far from being an ∞-groupoid: only the zero vector space is invertible.

4.2. $\mathcal{B}^{\oplus}\mathrm{O}$ in low dimensions

Using notation from Section 2.2.1, we will discuss explicitly the 1-, 2- and 3-simplices of $\mathcal{B}^{\oplus}\mathrm{O}$ for future reference, and leave higher simplices to the interested reader. For better readability, we will mostly not use the standard notation for face maps from Chapter 2, opting instead to indicate which vertices are included.

Warning 4.2.1. A 'vector space' is a point of $B\mathrm{O}^{\infty}_{\amalg}$.

1-**morphisms.** Let

$$F\colon \mathrm{Path}[1] \to B^{\oplus}\mathrm{O}$$

be a map of simplicial categories, i.e., a 1-simplex of $\mathcal{B}^{\oplus}\mathrm{O}$. Both objects $0, 1 \in [1]$ are sent to $*$. The mapping poset $P_{0,1}$ has the sole nontrivial element $\underline{01} := \{0, 1\} \in N_0(P_{0,1})$, the image of which determines F. Write

$$V_{01} := F(\underline{01}) \in \mathrm{Sing}_0 = \mathrm{Sing}_0 B\mathrm{O}^{\infty}_{\amalg},$$

so V_{01} is a vector space. In fact, F is determined by V_{01}.

More generally, for any $k \geq 1$, a k-path $\mathrm{Path}[k] \to \mathcal{C}$ in an arbitrary simplicial category $\mathcal{C}$ is determined by its values on simplices of dimensions $\leq k-1$ in the morphism spaces of $\mathrm{Path}[k]$, since the simplices of higher dimensions are degenerate.

2-**morphisms.** Let

$$F\colon \mathrm{Path}[2] \to B^{\oplus}\mathrm{O}$$

be a 2-simplex of $\mathcal{B}^{\oplus}\mathrm{O}$. Let $\iota_{ab}^{*}F\colon \mathrm{Path}[1] \to B^{\oplus}\mathrm{O}$ be the three faces, $\iota_{ab}\colon [1] \hookrightarrow [2]$ given by $0 \mapsto a$, $1 \mapsto b$ for $a < b$ in $[2]$, so that they are determined by vector spaces $V_{ab} = F(\underline{ab}) \in \mathrm{Sing}_0$ as above. The mapping poset

$$P_{0,2}^{\mathrm{op}} = \{\underline{012} \geq \underline{02}\}$$

includes two new pieces of information: a vector space $V_{012} = F(\underline{012}) \in \mathrm{Sing}_0$, and, seeing $\geq \in N_1(P_{0,2})$, a path $\gamma = F(\geq) \in \mathrm{Sing}_1$ with source V_{012} and target V_{02}.

Notice now that $\underline{012}$ is in the image of

$$P_{1,2}^{\mathrm{op}} \times P_{0,1}^{\mathrm{op}} \to P_{0,2}^{\mathrm{op}},$$

namely $\underline{012} = \underline{12} \cup \underline{01}$. As F is functorial, we have $V_{012} = V_{12} \oplus V_{01}$. Thus, F is determined by three vector spaces V_{01}, V_{12} and V_{02}, together with a path $\gamma\colon V_{12} \oplus V_{01} \to V_{02}$ in $B\mathrm{O}^{\infty}_{\amalg}$. Pictorially:

$$\begin{tikzcd}
 & * \arrow[dr, "V_{12}"] & \\
* \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right, "V_{12}\oplus V_{01}"'] \arrow[rr, phantom, bend right=20, "\Uparrow\gamma"] & & *
\end{tikzcd} \tag{4.2.2}$$

If V_{12} is the identity, i.e. $V_{12} = 0$, then this is just a path from V_{01} to V_{02}.

3-morphisms. Let

$$F\colon \mathrm{Path}[3] \to B^{\oplus}\mathrm{O}$$

be a 3-simplex of $\mathcal{B}^{\oplus}\mathrm{O}$. There are six non-degenerate edges giving vector spaces: $V_{ab} = F(\underline{ab})$, $0 \le a < b \le 3$. The four non-degenerate faces

$$\iota_{abc}\colon \mathrm{Path}[2] \hookrightarrow \mathrm{Path}[3] \to B^{\oplus}\mathrm{O}$$

are of the form of (4.2.2), which specifically is the face $d_3(F)$ corresponding to $(a,b,c) = (0,1,2)$. We have four paths of type $V_{abc} = V_{bc} \oplus V_{ab} \to V_{ac}$:

$$V_{12} \oplus V_{01} \to V_{02}, \quad V_{13} \oplus V_{01} \to V_{03}, \quad V_{23} \oplus V_{12} \to V_{13}, \quad V_{23} \oplus V_{02} \to V_{03}. \tag{4.2.3}$$

The mapping poset $P^{\mathrm{op}}_{0,3}$ is as follows:

$$\begin{tikzcd} & \underline{0123} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \underline{013} \arrow[dr] & & \underline{023} \arrow[dl] \\ & \underline{03} & \end{tikzcd}$$

The triangles above depict the two non-degenerate elements of $N_2(P^{\mathrm{op}}_{0,3})$, which F maps to $\mathrm{Sing}_2(B\mathrm{O}^{\infty}_{\mathrm{II}})$. That is, F gives homotopies filling these triangles as in

$$\begin{tikzcd} & V_{0123} \arrow[dl] \arrow[dr] \arrow[dd] & \\ V_{13} \oplus V_{01} \arrow[dr] \arrow[r, phantom, "\Longrightarrow"] & {} & V_{23} \oplus V_{02} \arrow[dl] \arrow[l, phantom, "\Longleftarrow"] \\ & V_{03} & \end{tikzcd} \tag{4.2.4}$$

The first and third paths of (4.2.3) give further decompositions of the sums on the left and right. Note the decompositions

$$V_{0123} = V_{123} \oplus V_{01} = V_{23} \oplus V_{012} = V_{23} \oplus V_{12} \oplus V_{01}. \tag{4.2.5}$$

Functoriality of F implies thereby that the path $V_{0123} \to V_{13} \oplus V_{01}$ in (4.2.4) is equal to $(V_{123} \to V_{13}) \oplus (V_{01} \xrightarrow{\mathrm{id}} V_{01})$ with the left summand given by (4.2.3); similarly, $V_{0123} \to V_{23} \oplus V_{02}$ is also already determined by (4.2.3), i.e., by $d_3(F)$. Analogously, we see that *all* 1-paths in (the image of) F are determined by its faces *except* for the path $V_{0123} \to V_{03}$.

Remark 4.2.6. If all but V_{01}, V_{02}, V_{03} are non-zero, then the right triangle of (4.2.4) reduces to

$$\begin{tikzcd} V_{03} \arrow[dd] \arrow[dr] & \\ {} & V_{02} \arrow[dl] \arrow[l, phantom, "\Longleftarrow"] \\ V_{01} & \end{tikzcd}$$

while the left triangle is degenerate.

4.3. The stratified Grassmannian

Our definition of the *stratified Grassmannian* is straightforward: it is the under-∞-category of $\mathcal{B}^{\oplus}\mathrm{O}$ of Definition 4.1.11 under its unique object $*$. (See Section 2.2.2.) It is that suggested in [7, Remark 2.7] except for the strictification of $B\mathrm{O}_{\mathrm{II}}$ into $B\mathrm{O}_{\mathrm{II}}^{\infty}$ and for being a quasi-category rather than a Segal space.

Definition 4.3.1. $\mathcal{V}^{\hookrightarrow} \coloneqq */\mathcal{B}^{\oplus}\mathrm{O}$.

Remark 4.3.2. A theorem of Lurie, [52, 01ZS], states that if $x \in \mathcal{C}$ is an object of a Kan-enriched category $\mathcal{C}$ and $x/\mathcal{C}$ is the simplicial under-category as defined in [52, 01Z8], then there is an equivalence of ∞-categories

$$\mathrm{N}^{\mathrm{hc}}(x/\mathcal{C}) \simeq x/\mathrm{N}^{\mathrm{hc}}(\mathcal{C})$$

if for every morphism $f\colon x \to y$ and every object $z \in \mathcal{C}$, pre-composition with f,

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z),$$

is a Kan fibration between Kan complexes.

This is *not* the case in our Kan-enriched category $B^{\oplus}\mathrm{O}$, since

$$- \oplus V\colon B\mathrm{O}_{\mathrm{II}}^{\infty} \to B\mathrm{O}_{\mathrm{II}}^{\infty} \tag{4.3.3}$$

is not a Kan fibration whenever $V \neq 0$. Moreover, $\mathcal{V}^{\hookrightarrow}$ is indeed not equivalent to $\mathrm{N}^{\mathrm{hc}}(*/B^{\oplus}\mathrm{O})$, as can be inferred by comparing their morphism spaces.

Indeed, Let $V \in B\mathrm{O}^m(l)$, $W \in B\mathrm{O}^{n+m}(k+l)$. The objects of $*/B^{\oplus}\mathrm{O}$ are the points of $B\mathrm{O}_{\mathrm{II}}^{\infty}$, and we have, by definition, that $\mathrm{Hom}_{*/B^{\oplus}\mathrm{O}}(V, W)$ is the ordinary fibre of (4.3.3) at W, so the subspace of $B\mathrm{O}^n(k)$ consisting of V' such that $V' \oplus V = W$. This is empty if, for instance, W is spanned by a $(k+l)$-frame in the second summand of $H^n \oplus H^m$. If non-empty, it is a singleton. Now, by results of Joyal and Lurie (see Hebestreit–Krause [45] for a direct proof), morphism spaces in homotopy-coherent nerves coincide, up to equivalence, with those of the original Kan-enriched category, thus

$$\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(*/B^{\oplus}\mathrm{O})}(V, W) \simeq \mathrm{Hom}_{*/B^{\oplus}\mathrm{O}}(V, W).$$

On the other hand, by [51, Lemma 5.5.5.12], $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(V, W)$ is the homotopy fibre of (4.3.3) at W, and so is equivalent to the space of paths in $B\mathrm{O}^{n+m}(k+l)$ that start at $V' \oplus V$ for some V' and end at W. In other words, we would have $\mathcal{V}^{\hookrightarrow} \simeq \mathrm{N}^{\mathrm{hc}}(*/B^{\oplus}\mathrm{O})$ *if* $B\mathrm{O}_{\mathrm{II}}^{\infty}$ were equipped with the discrete topology. This justifies Definition 4.3.1.

We will now explicate the morphisms of $\mathcal{V}^{\hookrightarrow}$ up to dimension 2. Via the identification $\Delta[0] \star \Delta[n] \simeq \Delta[n+1]$ as in Remark 2.2.10, n-simplices of $\mathcal{V}^{\hookrightarrow}$ are $(n+1)$-simplices $\mathcal{B}^{\oplus}\mathrm{O}$ with no qualification, which is to say that we have bijections

$$\mathcal{V}^{\hookrightarrow}_n \cong \mathrm{Fun}\left(\mathrm{Path}[n+1], B^{\oplus}\mathrm{O}\right),$$

since $\mathcal{B}^{\oplus}\mathrm{O}$ has a unique object. In particular, a 0-simplex of $\mathcal{V}^{\hookrightarrow}$ is a vector space V (recall Warning 4.2.1). Generalising Remark 2.2.11, the face and degeneracy maps of $\mathcal{V}^{\hookrightarrow}$ can be written in terms of those of $\mathcal{B}^{\oplus}\mathrm{O}$ as follows.

Lemma 4.3.4. $d_i^{\mathcal{V}^{\hookrightarrow}} = d_{i+1}^{\mathcal{B}^{\oplus}\mathrm{O}}$, $s_i^{\mathcal{V}^{\hookrightarrow}} = s_{i+1}^{\mathcal{B}^{\oplus}\mathrm{O}}$.

Proof. We have $(\mathrm{id}_0 \star \partial_i \colon \Delta[0] \star \Delta[k] \hookrightarrow \Delta[0] \star \Delta[k+1]) = (\partial_{1+i} \colon \Delta[k+1] \hookrightarrow \Delta[k+2])$ upon identifying $\Delta[0] \star \Delta[\bullet] \simeq \Delta[\bullet + 1]$, and similarly for the degeneracies. $\square$

Remark 4.3.5. A 1-morphism of $\mathcal{V}^{\hookrightarrow}$ is as in (4.2.2), with source V_{01} and target V_{02} (see Remark 2.2.11) together with a path, which we can summarise as $V_{01} \subseteq V_{12} \oplus V_{01} \xrightarrow{\gamma} V_{02}$. In this sense, morphisms of $\mathcal{V}^{\hookrightarrow}$ can be said to be 'injections of vector spaces' if one disregards γ. Taking constant γ, and using the inner product on H to choose the orthogonal complements canonically, includes proper vector space injections into the non-invertible morphisms of $\mathcal{V}^{\hookrightarrow}$. In view of Remark 4.3.2, identifying morphisms with injections amounts to equipping $B\mathrm{O}_{\mathrm{II}}^{\infty}$ with the discrete topology.

2-morphisms. We resume our exposition in dimension 2 before moving on to our next result. The purpose is to push the morphisms-as-injections point of view one dimension higher. It also serves to motivate the ideas in the proof of Theorem 4.3.11 below but can otherwise be skipped.

A 2-morphism of $\mathcal{V}^{\hookrightarrow}$ is a map

$$\gamma \colon \Delta[0] \star \Delta[2] \to \mathcal{B}^{\oplus}\mathrm{O}$$

whose edges may be described as follows:

$$\begin{tikzcd}
 & 2 \arrow[dr, "W_{23}"] & \\
1 \arrow[ur, "W_{12}"] \arrow[rr, "W_{13}"] & & 3 \\
 & 0 \arrow[ul, "V_{01}"] \arrow[uu, "V_{02}"] \arrow[ur, "V_{03}"'] &
\end{tikzcd} \tag{4.3.6}$$

We have the following three induced faces of γ:

$$\begin{tikzcd}
\Delta[0] \star \Delta[1] \arrow[r, hook, "\mathrm{id}\times\partial_i"] \arrow[d, "\simeq"] & \Delta[0] \star \Delta[2] \arrow[r, "\gamma"] \arrow[d, "\simeq"] & \mathcal{B}^{\oplus}\mathrm{O} \\
\Delta[2] \arrow[r, dashed, "\mathrm{id}\times\partial_i"'] \arrow[urr, bend right=40, "d_{i+1}^{\mathcal{B}^{\oplus}\mathrm{O}}\gamma"'] & \Delta[3] \arrow[ur, dashed, "\gamma"'] &
\end{tikzcd}$$

The $\{0,1\}$-edge of $\Delta[2]$ is called its *source edge*, and the $\{0,2\}$-edge of $\Delta[2]$ its *target edge*. We say, therefore, that the induced face $d_2^{\mathcal{B}^{\oplus}\mathrm{O}}\gamma$ is its *source face*, and $d_1^{\mathcal{B}^{\oplus}\mathrm{O}}\gamma$ its *target face*. These two faces share their respective source edges:

$$\begin{tikzcd}
\Delta[0]\star\Delta[0] \arrow[r, hook, "\mathrm{id}\times\partial_1"] \arrow[d, "\sim"] & \Delta[0]\star\Delta[1]\simeq\Delta[2] \arrow[r, "\gamma_2{,}\gamma_1"] & \mathcal{B}^{\oplus}\mathrm{O} \\
\Delta[1] \arrow[urr, dashed, "\gamma_{01}"'] & &
\end{tikzcd}$$

which is labelled by V_{01} in (4.3.6). The source face of γ is of type $V_{01} \subseteq W_{12} \oplus V_{01} \simeq V_{02}$, its target face of type $V_{01} \subseteq W_{13} \oplus V_{01} \simeq V_{03}$, and its intermediate face is of type $V_{02} \subseteq W_{23} \oplus V_{02} \simeq V_{03}$. Putting them together gives the picture

$$\begin{tikzcd}
V_{01} \subseteq W_{12}\oplus V_{01} \arrow[r] \arrow[dr, dash, "V_{01}\subseteq"] & V_{02} \subseteq W_{23}\oplus V_{02} \arrow[r] & V_{03} \\
& W_{13}\oplus V_{01} \arrow[ur] &
\end{tikzcd} \tag{4.3.7}$$

Consider now the final face of γ:

$$\begin{tikzcd}
\Delta[2] \arrow[r, hook, "\iota_1", "(2.2.6)"'] \arrow[rr, dashed, bend right, "\Gamma"'] & \Delta[0]\star\Delta[2] \arrow[r, "\gamma"] & \mathcal{B}^{\oplus}\mathrm{O}
\end{tikzcd}$$

It is of type $\Gamma = (W_{12} \subseteq W_{23} \oplus W_{12} \to W_{13})$. Concatenating the upper paths in (4.3.7) and inserting Γ gives

$$\begin{tikzcd}
V_{01} \arrow[r, dash, "\subseteq"] \arrow[dr, dash, "\subseteq"'] & W_{23}\oplus W_{12}\oplus V_{01} \arrow[r] \arrow[d, "\Gamma\oplus\mathrm{id}"] & V_{03} \\
& W_{13}\oplus V_{01} \arrow[ur] &
\end{tikzcd} \tag{4.3.8}$$

The left triangle clearly commutes. The homotopy in the left triangle of (4.2.4) (with all paths inverted) commutes the right triangle of (4.3.8) in view of (4.2.5).

Remark 4.3.9. If all but V_{01}, V_{02}, V_{03} are non-zero, then (4.3.8) reduces to

$$\begin{tikzcd}
V_{01} \arrow[r] \arrow[rr, bend right] & V_{02} \arrow[r] & V_{03}
\end{tikzcd}$$

together with a filler 2-path in $B\mathrm{O}^{\infty}_{\amalg}$.

The core of $\mathcal{V}^{\hookrightarrow}$. We have established above the sense in which the non-invertible morphisms of $\mathcal{V}^{\hookrightarrow}$ are given by proper injections of vector spaces. It is desirable, however, that $\mathcal{V}^{\hookrightarrow}$ contain no more invertible morphisms than the original infinite Grassmannians, so that no more information is added unnecessarily. Indeed, Theorem 4.3.11 below states just this.

Notation 4.3.10. For $\mathcal{C}$ an ∞-category, let $\mathcal{C}^{\simeq}$ denote its *maximal sub-∞-groupoid*, or *core*, i.e. the ∞-groupoid whose n-simplices are exactly those n-simplices of $\mathcal{C}$ whose edges are isomorphisms $\mathcal{C}$. This is indeed an ∞-groupoid by a result of Joyal [47]; see also [52, 019D]. We will write $\mathcal{V}^{\simeq} := (\mathcal{V}^{\hookrightarrow})^{\simeq}$.

We should note again that the second equivalence, from Remark 4.1.7, in the statement below, is very much non-canonical.

Theorem 4.3.11. $\mathcal{V}^{\simeq} \simeq B\mathrm{O}_{\amalg}^{\infty} \simeq * \amalg \mathbf{Z}_{+} \times B\mathrm{O}_{\amalg}^{+}$.

Proof. First, given a k-simplex γ of $B\mathrm{O}_{\amalg}^{\infty}$, we will construct a functor

$$\Gamma\colon \mathrm{Path}[k+1] \to B^{\oplus}\mathrm{O}$$

of simplicial categories. It is necessarily trivial on objects. Let now $i \leq j \in [k+1]$ and let

$$\alpha = (\alpha^0 \geq \cdots \geq \alpha^n) \in \mathrm{N}_n\left(P_{i,j}^{\mathrm{op}}\right)$$

with subposets $\alpha^x = \underline{\alpha_1^x, \ldots, \alpha_{n_x}^x}$, $\alpha_y^x \in [k+1]$, $\alpha_y^x < \alpha_{y'}^x$ strictly for $y < y'$, and $\alpha_1^x = i$, $\alpha_{n_x}^x = j$.

If $j = i$, then all such sequences trivial and each α^x consists of i alone, and therefore, by functoriality, $\Gamma(\alpha) = s_0^n(0) \in (B\mathrm{O}_{\amalg}^{\infty})_n$, the n-fold degenerate zero vector space. Let us therefore assume $i < j$.

If $i > 0$, we also set $\Gamma(\alpha) \coloneqq s_0^n(0)$.

If $i = 0$ and so $j > 0$ by assumption, every subposet α^x consists of at least two elements. Consider then the associated map

$$\begin{aligned} A\colon [n] &\to [k] \\ x &\mapsto \alpha_2^x - 1. \end{aligned}$$

It is functorial since the partial order $\leq$ is defined to mean that α is given by subsets of $[k+1]$ satisfying $\alpha^0 \supseteq \cdots \supseteq \alpha^n$, so $\alpha_2^{x'} \in \alpha^x$ and therefore $\alpha_2^x \leq \alpha_2^{x'}$ whenever $x \leq x'$. It is moreover well-defined since $\alpha_2^x - 1 \leq j - 1 \leq k$, and $\alpha_2^x - 1 > \alpha_1^x - 1 \geq 0$. Now, the rule

$$\begin{aligned} \Phi\colon \mathrm{N}_n\left(P_{i,j}^{\mathrm{op}}\right) &\to \Delta[k]_n = \mathrm{Hom}_{\mathbf{\Delta}}([n],[k]) \\ \alpha &\mapsto \Phi(\alpha) \coloneqq A \end{aligned}$$

is simplicial: let $\delta\colon [n'] \to [n]$ be a poset map, so $(\delta^*(\alpha))^x = \alpha^{\delta(x)}$ for $x \in [n']$, and observe that $\Phi(\delta^*\alpha)(x) = \alpha_2^{\delta(x)} - 1 = \delta^*(\Phi(\alpha))(x)$.

We thus obtain the maps

$$\begin{aligned} \Gamma\colon \mathrm{N}_n\left(P_{i,j}^{\mathrm{op}}\right) &\to (B\mathrm{O}_{\amalg}^{\infty})_n \\ \alpha &\mapsto \Phi(\alpha)^*\gamma \end{aligned}$$

for $n \geq 0$ which assemble into maps

$$\Gamma\colon \mathrm{Hom}_{\mathrm{Path}[k+1]}(i,j) \to B\mathrm{O}_{\amalg}^{\infty}$$

for all pairs $i \leq j$ in $[k+1]$. The simpliciality of Φ implies $\Gamma(\delta^*\alpha) = \delta^*\Gamma(\alpha)$, i.e., the simpliciality of Γ on morphism spaces.

We will now show that Γ is functorial. Let $\alpha \in \mathrm{N}_n\left(P_{i,j}^{\mathrm{op}}\right)$, $\beta \in \mathrm{N}_n\left(P_{j,l}^{\mathrm{op}}\right)$ be sequences as above, with $i \leq j \leq l$ in $[k+1]$, so

$$\beta \cup \alpha = \left(\beta^0 \cup \alpha^0 \geq \cdots \geq \beta^n \cup \alpha^n\right) \in \mathrm{N}_n\left(P_{i,l}^{\mathrm{op}}\right).$$

If $i = j = l$, then $\Gamma(\beta \cup \alpha) = s_0^n(0) \oplus s_0^n(0) = s_0^n(0)$; if $i = j < l$, then $\Phi(\beta \cup \alpha)(x) = (\beta^x \cup \alpha^x)_2 - 1 = \beta_2^x - 1$ since $\alpha^x = (\alpha_1^x)$ is degenerate and so

$(\beta^x \cup \alpha^x)_1 = \alpha_1^x = \beta_1^x$, hence $\Gamma(\beta \cup \alpha) = \Gamma(\beta) = \Gamma(\beta) \oplus s_0^n 0 = \Gamma(\beta) \oplus \Gamma(\alpha)$; if $i < j \leq l$, then analogously $\Phi(\beta \cup \alpha)(x) = \alpha_2^x - 1$, and so $\Gamma(\beta \cup \alpha) = \Gamma(\alpha) = \Gamma(\beta) \cup \Gamma(\alpha)$ because $j > 0$ gives $\Gamma(\beta) = s_0^n(0)$ by construction.

Let us observe that the maps

$$\begin{aligned} \Psi\colon (B\mathrm{O}^\infty_{\mathrm{II}})_k &\to \mathcal{V}^{\hookrightarrow}_k \\ \gamma &\mapsto \Psi(\gamma) \coloneqq \Gamma \end{aligned}$$

assemble into an ∞-functor $\Psi\colon B\mathrm{O}^\infty_{\mathrm{II}} \to \mathcal{V}^{\hookrightarrow}$. By Lemma 4.3.4, we must show that

$$\Psi(d_i\gamma) = d_{i+1}^{\mathcal{B}\oplus\mathrm{O}}(\Psi(\gamma)) \quad \text{and} \quad \Psi(s_i\gamma) = s_{i+1}^{\mathcal{B}\oplus\mathrm{O}}(\Psi(\gamma))$$

for all $i \in [k]$. For the first, we may assume $k \geq 1$, and take $j = 0$, $l > 0$, and $\alpha \in \mathrm{N}_n\left(P^{\mathrm{op}}_{0,l}\right) = \mathrm{Hom}_{\mathrm{Path}[k]}(0, l)$. The face map $\partial_i\colon [k-1] \hookrightarrow [k]$ composes with $\Phi(\alpha)\colon [n] \to [k-1]$ to give

$$\begin{aligned} [n] &\to [k] \\ x &\mapsto \begin{cases} \alpha_2^x - 1, & \alpha_2^x - 1 \leq i - 1, \\ \alpha_2^x, & \alpha_2^x - 1 \geq i. \end{cases} \\ &= \begin{cases} \alpha_2^x - 1, & \alpha_2^x \leq i, \\ \alpha_2^x, & \alpha_2^x \geq i + 1 \end{cases} \end{aligned}$$

so that $\Psi(d_i\gamma)(\alpha)$ is the pullback of γ along this map.

On the other hand, $d_{i+1}^{\mathcal{B}\oplus\mathrm{O}}$ is given by pre-composing with

$$\partial_{i+1}\colon \mathrm{Path}[k] \hookrightarrow \mathrm{Path}[k+1], \tag{4.3.12}$$

which on α reads $\partial_{i+1}(\alpha) = (\partial_{i+1}\alpha^0 \geq \cdots \geq \partial_{i+1}\alpha^n) \in \mathrm{Hom}_{\mathrm{Path}[k+1]}(0, \partial_{i+1}l)$ with $\partial_{i+1}\alpha^x = \underline{\partial_{i+1}(\alpha_1^x), \ldots, \partial_{i+1}(\alpha_{n_x}^x)} = \underline{0, \partial_{i+1}(\alpha_2^x), \ldots, \partial_{i+1}l}$ and so

$$\partial_{i+1}(\alpha)_2^x - 1 = \partial_{i+1}(\alpha_2^x) - 1 = \begin{cases} \alpha_2^x - 1, & \alpha_2^x \leq i, \\ \alpha_2^x, & \alpha_2^x \geq i + 1 \end{cases}.$$

By the above, we obtain $\partial_i\Phi(\alpha) = \Phi(\partial_{i+1}\alpha)$ and thus

$$\begin{aligned} \Psi(d_i\gamma)(\alpha) &= \Phi(\alpha)^*(d_i\gamma) = (\partial_i\Phi(\alpha))^*\gamma = (\Phi(\partial_{i+1}\alpha))^*\gamma \\ &= \partial_{i+1}^*\Phi(\alpha)^*\gamma \\ &= d_{i+1}^{\mathcal{B}\oplus\mathrm{O}}(\Psi(\gamma))(\alpha). \end{aligned}$$

Compatibility with degeneracies can be verified similarly.

We have thus defined an ∞-functor

$$\Psi\colon B\mathrm{O}^\infty_{\mathrm{II}} \to \mathcal{V}^{\simeq} \hookrightarrow \mathcal{V}^{\hookrightarrow},$$

which necessarily factors as depicted.

Let now

$$\sigma\colon \mathrm{Path}[k+1] \to B^{\oplus}\mathrm{O}$$

be a k-simplex of $\mathcal{V}^{\simeq}$, which is to say that the restrictions

$$\sigma|_{ijl}\colon \mathrm{Path}\{i \leq j \leq l\} \hookrightarrow \mathrm{Path}[k+1] \xrightarrow{\sigma} B^{\oplus}\mathrm{O},$$

as 1-morphisms of $\mathcal{V}^{\hookrightarrow}$, are isomorphisms. By Lemma 4.3.4, the relevant triples satisfy $i = 0$, $j \geq 1$. As discussed at around (4.2.2), $\sigma|_{0jl}$ is fully determined by a path in $BO_{\mathrm{II}}^{\infty}$ of type

$$\sigma(\underline{jl}) \oplus \sigma(\underline{0j}) \to \sigma(\underline{0l})$$

where $\sigma(\underline{0j})$ is the source of $\sigma|_{0jl}$ and $\sigma(\underline{0l})$ its target. However, since $\mathcal{V}^{\hookrightarrow}$ contains no morphism of type $W \to V$ if $\mathrm{rk}(W) > \mathrm{rk}(V)$, we conclude $\sigma(\underline{jl}) = 0$ since $\sigma|_{0jl}$ is an isomorphism. Since the pair $1 \leq j \leq l$ was arbitrary, this implies that for any $\alpha = \underline{\alpha_1, \ldots, \alpha_n} \in \mathrm{N}_0\left(P^{\mathrm{op}}_{j,l}\right)$ we have $\sigma(\alpha) = 0$ by decomposing $\alpha = \underline{\alpha_{n-1}\alpha_n} \cup \cdots \cup \underline{\alpha_1\alpha_0}$. Thus we obtain

$$\sigma(\underline{0, \alpha}) = \sigma(\underline{0, \alpha_1}) \tag{4.3.13}$$

by decomposing $\underline{0, \alpha} = \alpha \cup \underline{0, \alpha_1}$.

Now, as was noted in Remark 4.3.2, we have $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(V, W)$ is equivalent to the space of paths in $BO_{\mathrm{II}}^{\infty}$ that start $V' \oplus V$ for some V' and end at W, and we have shown that within $\mathcal{V}^{\simeq}$ this reduces to $V' = 0$, implying, for V and W in the same connected component of $BO_{\mathrm{II}}^{\infty}$, that $\mathrm{Hom}_{\mathcal{V}^{\simeq}}(V, W)$ is equivalent to the space of paths $V \to W$, which is exactly the morphism space of $BO_{\mathrm{II}}^{\infty}$ from V to W. Moreover, along the former equivalence, Ψ maps as the identity on morphism spaces. Since it is moreover a bijection on objects, we conclude (by [52, 01JX]) that it is an equivalence onto $\mathcal{V}^{\simeq}$ by virtue of being fully faithful and essentially surjective. □

Remark 4.3.14. Equation (4.3.13) may seem to lead to the following 'point-set' Ansatz to constructing an inverse to Ψ, with the goal of promoting $\mathcal{V}^{\simeq} \simeq BO_{\mathrm{II}}^{\infty}$ to an isomorphism. Namely, consider the sequence

$$\Lambda^k = \left(\underline{0, 1, 2, \ldots, k+1} \geq \underline{0, 2, 3, \ldots, k+1} \geq \cdots \geq \underline{0, k+1}\right) \in \mathrm{N}_k\left(P^{\mathrm{op}}_{0,k+1}\right).$$

Its image $\sigma(\Lambda^k) \in (BO_{\mathrm{II}}^{\infty})_k$ is of type

$$\sigma(\underline{0, 1}) \to \sigma(\underline{0, 2}) \to \cdots \to \sigma(\underline{0, k+1}).$$

The map $\Psi^{-1}\colon \mathcal{V}^{\simeq} \to BO_{\mathrm{II}}^{\infty}$, $\sigma \mapsto \sigma(\Lambda)$, however, is *not* necessarily simplicial. To compare faces, let $i \in [k]$ so that we have $\sigma(d_i\Lambda^k) = d_i(\sigma(\Lambda^k))$, while $(d^{\mathcal{B}^{\oplus}\mathrm{O}}_{i+1}\sigma)(\Lambda^{k-1}) = \sigma(\partial_{i+1}\Lambda^{k-1})$ by definition, so one might expect that $\sigma(d_i\Lambda^k) = \sigma(\partial_{i+1}\Lambda^{k-1})$, giving compatibility with face maps.[1] This need not hold: for instance, taking $k = 3$ and $i = 1$, this equation reads

$$\sigma(\underline{01234} \geq \underline{034} \geq \underline{04}) = \sigma(\underline{0134} \geq \underline{034} \geq \underline{04}).$$

While their vertices agree, the functoriality of σ does not necessitate that these two simplices agree.

Still, it is possible to give a map $\mathcal{V}^{\simeq} \to BO_{\mathrm{II}}^{\infty}$ using an idea to be developed later. Namely, to σ we can associate its value on a certain topological k-simplex $\bigtriangledown \subseteq \left|\mathrm{N}_{\bullet}\left(P^{\mathrm{op}}_{0,k+1}\right)\right|$ which, after passing through the adjunction of geometric realisation with the singular chains functor, yields a k-simplex of $BO_{\mathrm{II}}^{\infty}$. We get back to this in Remark 5.2.53, using the construction of $\bigtriangledown$ in Section 5.2.4.

[1]Here, ∂_{i+1} applies to Λ^{k-1} as in (4.3.12).

The following suggests $*/\mathrm{N}^{\mathrm{hc}}(-)$ as a means to adjoin non-invertible paths to a smooth collection of objects with a monoidal structure.

Corollary 4.3.15. *Let M be a topological monoid whose only invertible element is its unit. Then*

$$(*/\mathrm{N}^{\mathrm{hc}}(BM))^{\simeq} \cong M$$

Moreover, $(*/\mathrm{N}^{\mathrm{hc}}(BM)) \simeq \mathrm{N}^{\mathrm{hc}}(*/BM)$ if and only if M is discrete.

Proof. The proof of Theorem 4.3.11 applies mutatis mutandis. For the second statement, see Remarks 4.3.2 and 4.3.5, which also apply for the same reasons. $\square$

Remark 4.3.16. This is not to say that $*/-$ and N^{hc} commute if and only if the argument simplicial category is discrete, but only that this happens to be the case in the situation of Corollary 4.3.15.

CHAPTER 5

The unpacking map

Our next goal is to construct a map

$$\mathbf{U} : \mathcal{EX}(BO(n,m)) \to \mathcal{V}^{\hookrightarrow}$$

from the exit path ∞-category of the (n,m)-Grassmannian of Example 3.2.17 to the stratified Grassmannian. The theory of tangential structures on linked spaces will then be able to interface with the conically-smooth variant.

5.1. The unpacking map in low dimensions

The map restricted to $BO(n)_\bullet$ and $BO(n+m)_\bullet$ inside $\mathcal{EX} \coloneqq \mathcal{EX}(BO(n,m))$ is defined to be inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$. It remains to define the restriction

$$\mathcal{EX}_{k+1} \supset \mathcal{P}_k^{\Delta} \to \mathcal{V}_{k+1}^{\hookrightarrow} \cong \mathrm{Fun}\left(\mathrm{Path}[k+2], B^{\oplus}\mathrm{O}\right),$$

for $k \geq 0$. We will explain dimensions 1 and 2 verbosely before giving the full construction without further motivation.

An element $(\gamma, 1)$ of $\mathcal{P}_0^{\Delta}$ – the exit index in this dimension is necessarily 1 – corresponds to a path γ in $BO(n+m)$ whose initial point is a direct sum $V_{12} \oplus V_{01}$ with $V_{01} \in BO(n)$, $V_{12} \in BO(m)$. Denoting the endpoint by V_{02}, γ determines a 2-path by arranging the data exactly as in (4.2.2). We have thus defined

$$\mathbf{U}|_{\leq 1} \colon \mathcal{EX}_{\leq 1} \to \mathcal{V}^{\hookrightarrow}. \tag{5.1.1}$$

Lemma 5.1.2. *The assignment* $\mathbf{U}|_{\leq 1}$ *is functorial, i.e., compatible with all relevant face and degeneracy maps.*

Proof. The source of the image of γ is V_{01}, and the target is V_{02}, which are, by construction, the images of the source and target of γ in $\mathcal{EX}$, respectively:

$$d_1^{\mathcal{V}^{\hookrightarrow}}(\mathbf{U}(\gamma,1)) = V_{01} = \mathbf{U}\left(\mathrm{pr}\left(d_1^{BO(n+m)}\gamma\right)\right) = \mathbf{U}\left(d_1^{\mathcal{EX}}(\gamma,1)\right),$$

and

$$d_0^{\mathcal{V}^{\hookrightarrow}}(\mathbf{U}\gamma) = V_{02} = \mathbf{U}\left(d_0^{BO(n+m)}\gamma\right) = \mathbf{U}\left(d_0^{\mathcal{EX}}(\gamma,1)\right).$$

Compatibility with degeneracies is immediate since in these dimensions there are no degenerate non-invertible paths. $\square$

Theorem 5.1.3. *The assignment* $\mathbf{U}|_{\leq 1}$ *of* (5.1.1) *extends to an* ∞*-functor*

$$\mathbf{U} \colon \mathcal{EX}(BO(n,m)) \to \mathcal{V}^{\hookrightarrow}.$$

We call $\mathbf{U}$ the *unpacking map.* Such an extension involves, as we will see, some contractible choices. Somewhat mysteriously, we will see that none of these choices breaks functoriality. Our construction will be formulated as an inductive proof of existence. First, we will discuss what $\mathbf{U}$ has to do with 2-morphisms at a phenomenological level so as to elucidate the essential issues to be overcome.

Construction 5.1.4. Assignment (5.1.1), $\mathbf{U}|_{\leq 1}\colon \mathcal{EX}_{\leq 1} \to \mathcal{V}^{\hookrightarrow}$, extends by functoriality to a subset of simplicies of $\mathcal{EX}$ in every dimension. That is, for any finite-order simplicial operator $\mathcal{O} = \prod \alpha_i$, $\alpha_i = s_{j_i}$ or $d_{j_i'}$ for any collection of indices j, j' such that the application makes sense, we set $\mathbf{U}(\mathcal{O}(\gamma, 1)) \coloneqq \mathcal{O}\mathbf{U}(\gamma, 1)$ for any $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$. Lemma 5.1.2 states exactly that this is well-defined. In effect, this gives a new definition only on degenerate simplices of dimension higher than 1 stemming from exit 1-paths, so we could have considered $\alpha = s_{j_i}$ only. This defines $\mathbf{U}$ on the simplicial subset of $\mathcal{EX}$ generated by $B\mathrm{O}(n)$, $B\mathrm{O}(n+m)$, and $\mathcal{P}_0^{\Delta} \subset \mathcal{EX}_1$. We write $\overline{\mathcal{EX}_{\leq 1}}$ for this simplicial subset, and again

$$\mathbf{U}_{\leq 1}\colon \overline{\mathcal{EX}_{\leq 1}} \to \mathcal{V}^{\hookrightarrow}$$

for the resulting ∞-functor.

We will observe a filtration $\bigcup \overline{\mathcal{EX}_{\leq k}} = \mathcal{EX}$, whereupon it will suffice to extend $\mathbf{U}$ along it. This will be our strategy to prove Theorem 5.1.3.

5.1.1. 2-morphisms. Exit paths in $\mathcal{P}_k^{\Delta}$ come in $k+1$ classes according to their exit indices, which need to be mapped to $\mathcal{V}^{\hookrightarrow}$ in different ways.

First, in order to uniquely determine a 3-path in $B^{\oplus}\mathrm{O}$, it is enough to map out of the sets $N_{\leq 2}(P_{i,j}^{\mathrm{op}})$ into $B\mathrm{O}_{\leq 2} \coloneqq (B\mathrm{O}_{\mathrm{II}}^{\infty})_{\leq 2}$, since higher dimensions are degenerate. Indeed, in general, a κ-path in $B^{\oplus}\mathrm{O}$ is determined in dimensions $\leq \kappa - 1$.

Notation 5.1.5. In the rest of this work, we write $B\mathrm{O} = B\mathrm{O}_{\mathrm{II}}^{\infty}$ at the risk of confusion with common K-theoretical notation.

Definition 5.1.6. We call those 1-morphisms in Path$[l]$ that are of type

$$\mathrm{N}_0(P_{\alpha\beta}^{\mathrm{op}}) \ni \underline{\alpha\beta} \coloneqq \{\alpha < \beta\} \subset [l]$$

simple, as they are simple with respect to $\cup$. The remaining morphisms we call *composite*. Similarly for higher morphisms.

Example 5.1.7. For instance, in $N_1(P_{1,4}^{\mathrm{op}})$, $(\underline{1234} > \underline{124}) = (\underline{234} \cup \underline{12} > \underline{24} \cup \underline{12}) = (\underline{234} > \underline{24}) \cup (\underline{12} > \underline{12})$ is composite, but $\underline{1234} > \underline{14}$ is simple. More generally, any arrow with target a pair is simple, and the others are composite.

The simple 1-morphisms in Path[3] are of type $\underline{\alpha\beta} \subset [3]$, which by a 3-path are mapped to $V_{\alpha\beta} \in B\mathrm{O}_0$. When $\beta = \alpha+2$ (of which type there are two pairs), there are arrows $\underline{\alpha, \alpha+1, \beta} = \underline{\alpha+1, \beta} \cup \underline{\alpha, \alpha+1} > \underline{\alpha\beta}$ in $N_1(P_{\alpha\beta}^{\mathrm{op}})$, which determine paths $V_{\alpha+1,\beta} \oplus V_{\alpha,\alpha+1} \to V_{\alpha\beta}$, i.e., $V_{12} \oplus V_{01} \to V_{02}$ and $V_{23} \oplus V_{12} \to V_{13}$ in $B\mathrm{O}_1$, namely two of the face 2-paths. The remaining two faces are supplied analogously by considering $(\alpha, \beta) = (0, 3)$ and the compositions $\underline{013} = \underline{13} \cup \underline{01}$ and $\underline{023} = \underline{23} \cup \underline{02}$. Finally, again for $(\alpha, \beta) = (0, 3)$, consider $\underline{0123} = \underline{23} \cup \underline{12} \cup \underline{01}$, which is to be mapped to $V_{0123} = V_{23} \oplus V_{12} \oplus V_{01}$. Out of $N_1(P_{0,3}^{\mathrm{op}})$ we receive paths $V_{0123} \to V_{03}$, $V_{0123} \to V_{013}$, $V_{0123} \to V_{023}$. The two non-degenerate elements $(\underline{0123} > \underline{023} > \underline{03})$ and $(\underline{0123} > \underline{013} > \underline{03})$ in $N_2(P_{0,3}^{\mathrm{op}})$ are to map in $B\mathrm{O}_2$ to

$$\begin{tikzcd} & V_{023} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} = \begin{tikzcd} & V_{23} \oplus V_{02} \arrow[dr] & \\ V_{23} \oplus V_{12} \oplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{5.1.8}$$

and

$$\begin{tikzcd} & V_{013} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} = \begin{tikzcd} & V_{13} \oplus V_{01} \arrow[dr] & \\ V_{23} \oplus V_{12} \oplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{5.1.9}$$

We have thus summed up the data needed to provide a functor $\mathrm{Path}[3] \to B^{\oplus}\mathrm{O}$.

Now, let us start with paths of exit index $2 \in \{1, 2\}$. Such an exit path $(\gamma, 2)$ (in $\mathcal{P}_1^{\Delta} \subset \mathcal{EX}_2$) consists of a 2-simplex $\gamma \in B\mathrm{O}(n+m)_2$ of type

$$\begin{tikzcd} & K \\ W \oplus V \arrow[ur, "\gamma_{\oplus}"] \arrow[r, "\gamma_W \oplus \gamma_V"'] & W' \oplus V' \arrow[u, "\gamma_{\oplus'}"'] \end{tikzcd} \tag{5.1.10}$$

where the bottom edge comes from $B\mathrm{O}(n) \times B\mathrm{O}(m)$ (whence it is $\oplus$ of two paths). If $(\gamma, 2)$ is in $\overline{\mathcal{EX}_{\leq 1}}$, then $\mathbf{U}(\gamma, 2)$ is already defined by Construction 5.1.4. Let us assume, therefore, that $(\gamma, 2)$ is *not* degenerate. The natural choice for the image, visualised as a 3-simplex of $\mathcal{B}^{\oplus}\mathrm{O}$, is

$$\mathbf{U}(\gamma, 2) = \begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"] & & 3 \\ & 0 \arrow[ul, "V"] \arrow[uu, "V'"] \arrow[ur, "K"'] & \end{tikzcd} .$$

Indeed, the edges in (5.1.10) supply the face triangles. The fact that the bottom edge is of type $\gamma_W \oplus \gamma_V$ is crucial, since the summand paths supply the triangles adjacent to the edge decorated by the zero vector space. The only wrinkle is that the upper face requires a path $W' \to W$, which can be taken to be the (standard) inverse of γ_W, which we will denote by γ_W^{-1}.[1] As for (5.1.8),

$$\begin{tikzcd} & W' \oplus V' \arrow[dr, "\gamma_{\oplus'}"] & \\ W' \oplus V \arrow[ur, "\text{(i)}"] \arrow[rr, "\text{(ii)}"'] & & K \end{tikzcd} , \tag{5.1.11}$$

[1]The meaning of inversion, in any dimension, is recalled in Notation 5.2.4.

note that we must yet choose the paths $W' \oplus V \to W' \oplus V'$ and $W' \oplus V \to K$ (corresponding to the arrows $\underline{0123} > \underline{023}$ and $\underline{0123} > \underline{03}$). To this end, consider the diagram

$$\begin{tikzcd}
W' \oplus V' \arrow[r, "\gamma_{\oplus'}"] & K \\
W' \oplus V \arrow[u, "\mathrm{id}_{W'} \oplus \gamma_V"] \arrow[r, "\gamma_W^{-1} \oplus \mathrm{id}_V"'] & W \oplus V \arrow[u, "\gamma_{\oplus}"'] \arrow[ul, "\gamma_W \oplus \gamma_V"]
\end{tikzcd}$$

and choose the obvious fillers. By id_A we mean the constant (degenerate) loop $s_0 A$ at A. This suggests using (i) $= \mathrm{id}_{W'} \oplus \gamma_V$, (ii) $= (\gamma_W^{-1} \oplus \mathrm{id}_V) * \gamma_{\oplus}$ (we write concatenation from left to right) whereupon the obvious filler can be chosen. Similarly, for (5.1.9), i.e.,

$$\begin{tikzcd}
 & W \oplus V \arrow[dr, "\gamma_{\oplus}"] & \\
W' \oplus V \arrow[ur, "(\mathrm{i})'"] \arrow[rr, "(\mathrm{ii})'"'] & & K
\end{tikzcd} \quad , \tag{5.1.12}$$

consider

$$\begin{tikzcd}
W \oplus V \arrow[r, "\gamma_{\oplus}"] & K \\
W' \oplus V \arrow[u, "\gamma_W^{-1} \oplus \mathrm{id}_V"] \arrow[r, "\mathrm{id}_W \oplus \gamma_V"'] & W' \oplus V' \arrow[u, "\gamma_{\oplus'}"'] \arrow[ul, "\gamma_W^{-1} \oplus \gamma_V^{-1}"]
\end{tikzcd}$$

and proceed similarly. This completes the construction of $N_{\leq 2}(P_{0,3}^{\mathrm{op}}) \to B\mathrm{O}_{\leq 2}$ and so of the 3-path $\mathrm{Path}[3] \to B^{\oplus}\mathrm{O}$ associated to the exit path $(\gamma, 2)$.

The image of an index-1 exit path

$$\begin{tikzcd}
K \arrow[r, "\gamma_K"] & K' \\
W \oplus V \arrow[u, "\gamma_{\oplus}"] \arrow[ur, "\gamma_{\oplus'}"']
\end{tikzcd} \tag{5.1.13}$$

is constructed analogously, with its picture as a 3-simplex of $\mathcal{B}^{\oplus}\mathrm{O}$ given by

$$\begin{tikzcd}
 & 2 \arrow[dr, "0"] & \\
1 \arrow[ur, "W"] \arrow[rr, "W"] & & 3 \\
 & 0 \arrow[ul, "V"] \arrow[uu, "K"] \arrow[ur, "K'"'] &
\end{tikzcd} \quad .$$

We have thus defined $\mathcal{EX}_{\leq 2} \to \mathcal{V}_{\leq 2}^{\rightarrow}$. (A systematic account will follow in Section 5.2). Writing

$$\overline{\mathcal{EX}_{\leq 2}} \coloneqq \overline{(\mathcal{EX}_2 \smallsetminus \overline{\mathcal{EX}_{\leq 1}}) \cup \overline{\mathcal{EX}_{\leq 1}}}$$

for the simplicial subset of $\mathcal{EX}$ generated by $\overline{\mathcal{EX}_{\leq 1}}$ together with the elements of $\mathcal{P}_1^{\Delta}$ that were not in $\overline{\mathcal{EX}_{\leq 1}}$, we claim that Construction 5.1.4 applies mutatis mutandis to yield an ∞-functor

$$\mathbf{U}_{\leq 2} \colon \overline{\mathcal{EX}_{\leq 2}} \to \mathcal{V}^{\hookrightarrow},$$

so we must provide an analogoue of Lemma 5.1.2. In particular, this will show that the contractible choices we have made along the way have had no bearing on functoriality.

Lemma 5.1.14. *The map* $\mathbf{U}_{\leq 2}$ *is a well-defined extension of* $\mathbf{U}_{\leq 1}$.

Proof. Consider again an index-2 exit path $(\gamma, 2)$ as in (5.1.10). Its source edge is the path $d_2^{\mathcal{EX}}(\gamma, 2) = (\gamma_V \colon V \to V') \in B\mathrm{O}_1 \subset \mathcal{EX}_1$. Since its two remaining edges are vertical, they are the elements of $\mathcal{P}_0^{\Delta}$ induced by $\gamma_{\oplus'}$ and $\gamma_{\oplus}$. Now, recall Lemma 4.3.4 that $\mathbf{U}(\gamma, 2)$ is identified with an element of $\mathcal{B}^{\oplus}\mathrm{O}_3 \cong \mathcal{V}_2^{\hookrightarrow}$ via $\Delta[0] \star \Delta[2] \simeq \Delta[3]$. Accordingly, face maps apply on the factor $\Delta[2]$, i.e., ∂ acts as $\mathrm{id}_{\Delta[0]} \star \partial$. In the picture in $\mathcal{B}^{\oplus}\mathrm{O}_3$, this means that when pulling back along a face map $\partial \colon \Delta[1] \hookrightarrow \Delta[2]$, we restrict to the triangle whose top edge is specified by ∂; e.g., $\partial_2 \colon \Delta[1] \hookrightarrow \Delta[2]$, which skips 2, applies to give

$$d_2^{\mathcal{V}^{\hookrightarrow}} \mathbf{U}(\gamma, 2) = \begin{tikzcd} & 1 \arrow[dr, "0"] & \\ 0 \arrow[ur, "V"] \arrow[rr, "V'"'] & & 2 \end{tikzcd},$$

which is precisely $\mathbf{U}(\gamma_V)$. As for the compatibility with degeneracies, there is, by construction, nothing to show. We have $s_i \mathbf{U}_{\leq 2}(\gamma, 2) = \mathbf{U}_{\leq 2}(s_i(\gamma, 2))$ since $\mathbf{U}_{\leq 2}(s_i(\gamma, 2)) = s_i \mathbf{U}_{\leq 2}(\gamma, 2)$ by construction, and this is well-defined *if* $s_i(\gamma, 2) \notin \overline{\mathcal{EX}_{\leq 1}}$, so that $\mathbf{U}_{\leq 2}$ does not clash with $\mathbf{U}_{\leq 1}$. But $s_i(\gamma, 2) = \mathcal{O}(\gamma', 1)$ would imply

$$(\gamma, 2) = d_i s_i(\gamma, 2) = d_i \mathcal{O}(\gamma', 1) = \mathcal{O}'(\gamma', 1) \in \overline{\mathcal{EX}_{\leq 1}}, \tag{5.1.15}$$

which is precluded. This shows that $\mathbf{U}_{\leq 2}$ is a well-defined extension of $\mathbf{U}_{\leq 1}$. We leave the analogous treatment of the remaining two faces and of the index-1 case to the reader. $\square$

Remark 5.1.16. So as to avoid confusion, note in particular that the top face

$$\begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"'] & & 3 \end{tikzcd},$$

given by γ_W^{-1} is *not* a face in $\mathcal{V}^{\hookrightarrow}$, nor is $\gamma_W \in B\mathrm{O}(m)_1 \subset \mathcal{EX}_1$ a face of $(\gamma, 2)$ in $\mathcal{EX}$.

Remark 5.1.17. We *had to* assume that $(\gamma, 2)$ is non-degenerate in order to apply the construction above, or else we would have broken functoriality. For instance, suppose the exit 2-path in question is degenerate: $(\gamma, 2) = s_0(\gamma', 1) = (s_0\gamma', 2)$. It can be easily verified that the construction above applied to this would yield (i) = id and (ii) = id $* \gamma'$ in (5.1.8), whereas the counterparts of these edges in $s_0(\mathbf{U}(\gamma', 1))$ are id and id, respectively. These, as well as the filler 2-simplices, are *not* necessarily the same. There does not seem to be a natural closed-form formula for $\mathbf{U}$, hence our inductive-recursive construction.

Remark 5.1.18. That, e.g., a 2-path $x \in \mathcal{EX}_2$ is *not* in $(\overline{\mathcal{EX}_{\leq 1}})_2 \subseteq \mathcal{EX}_2$ is equivalent to it not being degenerate in the usual sense. Indeed, say $x = \mathcal{O}y$ for some simplicial operator $\mathcal{O}$ and $y \in \mathcal{EX}_{0/1}$. Then $\mathcal{O}$ must contain at least one degeneracy, since otherwise y would be in $\mathcal{EX}_{\geq 3}$ or $y = x$ already. So let s be the last (left-most) degeneracy in $\mathcal{O}$ and move it through the face maps in $\mathcal{O}$ to the left of s using simplicial identities, so that $\mathcal{O} = s'\mathcal{O}'$ for a resulting degeneracy s'. Then $x = s'\mathcal{O}'y$ is degenerate.

We will note the proof for completeness. Manifestly, it applies in any dimension.

Proof of Remark 5.1.18. One can use $d_\alpha s_\beta = s_{\beta-1} d_\alpha$ for $\alpha < \beta$, $d_\alpha s_\beta = \mathrm{id}$ for $\alpha \in \{\beta, \beta+1\}$ or $d_\alpha s_\beta = s_\beta d_{\alpha-1}$ for $\alpha > \beta + 1$. After using the latter, move on to the next degeneracy in $\mathcal{O}$, and iterate. If there is none, then we are again in the situation where $\mathcal{O}$ only contains face maps so that $y \in \mathcal{EX}_{\geq 3}$, which is absurd. $\square$

Still, we will keep using these simplicial closures for convenience, as they simplify some arguments – notably the latter part of the proof of Lemma 5.1.14, which concerns degeneracies. It is valid in any dimension.

5.2. The proof of Theorem 5.1.3

Now we proceed to give the general construction. We will first give a systematic account of

$$\mathcal{P}_1^\Delta \to \big[\mathrm{Path}[3], B^\oplus \mathrm{O}\big]$$

in such a way that the ideas generalise to all dimensions. It will be conventient to slightly rearrange the visual representation of exit paths.

Notation 5.2.1. For $(\gamma, 1) \in \mathcal{P}_0^\Delta$, the diagram $W \oplus V \to K$ depicts $\gamma \in B\mathrm{O}_1$. Instead, $V \xrightarrow{(W,\gamma)} K$ or $V \xrightarrow{W} K$ for short, depicts $(\gamma, 1)$ more informatively. Similarly, we sometimes depict $(\gamma, 2) \in \mathcal{P}_1^\Delta$ by
$$\begin{tikzcd} & K \\ V \arrow[ur, "W"] \arrow[r] & V' \arrow[u, "W'"'] \end{tikzcd}$$
, etc.

Notation. $[A, B] \coloneqq \mathrm{Fun}(A, B)$.

Notation 5.2.2. We write $\mathfrak{N}\colon B\mathrm{O}(n) \times B\mathrm{O}(m) \to B\mathrm{O}(m)$ to denote the second coordinate projection. When we apply $\mathfrak{N}$ to a low face of an exit path (γ, j), we mean that, first, the corresponding face of γ is to be taken, which is then (unambiguously) to be identified with a simplex of the link $B\mathrm{O}(n) \times B\mathrm{O}(m)$, and then $\mathfrak{N}$ is to be applied. Namely, we have, by abuse of notation, a map

$$\mathfrak{N}\colon \mathcal{P}_*^\Delta \to B\mathrm{O}(m)_*$$

for each $* \geq 0$, given by the composition

$$\begin{tikzcd} \mathcal{P}^{\Delta}_* \arrow[r, "(\gamma,j)\mapsto\Gamma_j=\gamma\circ\mathcal{C}_j"] \arrow[rrr, dashed, bend right, "\mathfrak{N}"'] & \mathcal{P}_* \arrow[r, two heads, "\mathbf{s}"] & \mathcal{L}_* \arrow[r, "\mathfrak{N}"] & BO(m)_* \end{tikzcd}$$

where we have not omitted $*$ since $\mathcal{P}^{\Delta}$ is not a simplicial set. Note that the result is degenerate unless the exit index is maximal.

Definition 5.2.3. The maximal low sub-simplex of $(\gamma, j) \in \mathcal{P}^{\Delta}_k$ is (the image under π of) $\gamma|_{0,\ldots,j-1}$. In $\mathcal{EX}(BO(n,m))$, this means $\gamma|_{0,\ldots,j-1} \in (BO(n) \times BO(m))_{j-1}$,[2] and so we write

$$\mathfrak{N}(\gamma, j) \coloneqq \mathrm{pr}_2(\gamma|_{0,\ldots,j-1}) \in BO(m)_{j-1}$$

and call it the *normal component* of (γ, j).

Notation 5.2.4. For X a space, we denote by Op the canonical isomorphism $X \simeq X^{\mathrm{op}}$ of Kan complexes, by which we mean $\mathrm{Sing}_\bullet(X) \simeq \mathrm{Sing}_\bullet(X)^{\mathrm{op}}$. It is given by inverting simplices by pulling back along the maps

$$\mathrm{Op}\colon \Delta^n \to \Delta^n,\ (a_0, \ldots, a_n) \mapsto (a_n, \ldots, a_0)$$

of the standard topological n-simplex, $n \geq 0$. See e.g. [52, 003R]. For example, when we wrote γ^{-1} above and called it the 'canonical inverse', this meant $\gamma^{-1} = \mathrm{Op}(\gamma) = \left(\Delta^1 \xrightarrow{\mathrm{Op}} \Delta^1 \xrightarrow{\gamma} X\right)$, so that $d_0\gamma^{-1} = d_1\gamma$, $d_1\gamma^{-1} = d_0\gamma$. More generally, for any simplicial set S, we have $S^{\mathrm{op}}_* = S_*$, and, in dimension n, $d_i^{S^{\mathrm{op}}} = d^S_{n-i}$, $s_i^{S^{\mathrm{op}}} = s^S_{n-i}$.

Notation 5.2.5. Let $\alpha_0, \ldots, \alpha_\ell \in [k]$. By $\mathrm{Path}[\alpha_0, \ldots, \alpha_\ell] \cong \mathrm{Path}[\ell]$ we denote the full simplicial subcategory of $\mathrm{Path}[k]$ generated by the objects $\alpha_0, \ldots, \alpha_\ell$.

5.2.1. $\mathcal{P}^{\Delta}_0 \to [\mathrm{Path}[2], B^{\oplus}\mathrm{O}]$. We define $\mathbf{U}$ hereon by

$$(\gamma, 1) = \begin{array}{c} K \\ \gamma\big\uparrow \\ W \oplus V \end{array} \mapsto \begin{cases} \mathrm{N}_0(P^{\mathrm{op}}_{0,a}) \to BO_0, & \underline{0,a} \mapsto \{a-1\}^*(\gamma, 1) = \begin{cases} V, & a = 1 \\ K, & a = 2 \end{cases} \\ \mathrm{N}_0(P^{\mathrm{op}}_{1,2}) \to BO_0, & \underline{12} \mapsto \mathrm{Op}\mathfrak{N}(\gamma, j) = \mathfrak{N}(\gamma, j) = W \\ \mathrm{N}_1(P_{0,2}) \to BO_1, & (\underline{012} > \underline{02}) \mapsto \gamma. \end{cases}$$

This induces $\mathbf{U}_{\leq 1}\colon \overline{\mathcal{EX}_{\leq 1}} \to \mathcal{V}^{\hookrightarrow}$ as in Construction 5.1.4.

5.2.2. $\mathcal{P}^{\Delta}_1 \to [\mathrm{Path}[3], B^{\oplus}\mathrm{O}]$. While skipping to the induction step below is now possible, we will treat this dimension explicitly in order to illustrate the ideas. Let $(\gamma, j) \in \mathcal{P}^{\Delta}_1$ and assume $(\gamma, j) \notin \overline{\mathcal{EX}_{\leq 1}}$.

Induced faces. We first define the faces of $\mathbf{U}(\gamma, j)$. The faces $d^{\mathcal{V}^{\hookrightarrow}}_{0,1,2}\mathbf{U}(\gamma, 2)$ are defined by $\mathbf{U}_{\leq 1}$ via functoriality, i.e., by

$$d_i^{\mathcal{V}^{\hookrightarrow}}(\mathbf{U}_{\leq 2}(\gamma, j)) \coloneqq \mathbf{U}_{\leq 1}(d_i^{\mathcal{EX}}(\gamma, j)). \tag{5.2.6}$$

[2] We do not distinguish $BO(n) \times BO(m)$ and its image in $BO(n+m)$ in notation.

This fixes $\mathbf{U}_{\leq 2}(\gamma, j)$ by Lemma 4.3.4 on the subcategories $\mathrm{Path}[0, k, l] \cong \mathrm{Path}[2]$ of $\mathrm{Path}[3]$, for $1 \leq k < l \leq 3$ (Notation 5.2.5). This is consistent due to the well-definedness of $\mathbf{U}_{\leq 1}$ as shown in Lemma 5.1.2.

The top face. The remaining $\mathcal{B}^{\oplus}\mathrm{O}$-face $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}(\mathbf{U}(\gamma, j))$ is the restriction to the full simplicial subcategory $\mathrm{Path}[1, 2, 3]$. Any pair $1 \leq \alpha < \beta \leq 3$ specifies a restriction along $\mathrm{Path}[\alpha, \beta] \hookrightarrow \mathrm{Path}[1, 2, 3]$, and writing δ for the remaining element of $\{1, 2, 3\}$, these restrictions must by functoriality coincide with

$$\begin{aligned} d_{\delta-1}^{\mathcal{B}^{\oplus}\mathrm{O}} d_0^{\mathcal{B}^{\oplus}\mathrm{O}}(\mathbf{U}(\gamma, j)) &= d_0^{\mathcal{B}^{\oplus}\mathrm{O}} d_\delta^{\mathcal{B}^{\oplus}\mathrm{O}}(\mathbf{U}(\gamma, j)) \\ &= d_0^{\mathcal{B}^{\oplus}\mathrm{O}} d_{\delta-1}^{\mathcal{V}^{\hookrightarrow}}(\mathbf{U}(\gamma, j)) \\ &\overset{(5.2.6)}{=} d_0^{\mathcal{B}^{\oplus}\mathrm{O}} \mathbf{U}_{\leq 1}(d_{\delta-1}^{\mathcal{E}\mathcal{X}}(\gamma, j)) \end{aligned}$$

where we used the simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$ for the first equation and Lemma 4.3.4 for the second. In other words, the edges of the $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$-face are determined already by $\mathbf{U}_{\leq 1}$. Therefore, only the restriction to the $(2-1=1)$-dimensional simplices of $\mathrm{N}_\bullet(P_{1,3}^{\mathrm{op}})$, that is,

$$\begin{aligned} \mathrm{N}_1(P_{1,3}^{\mathrm{op}}) &\to B\mathrm{O}_1 \\ (\underline{123} = \underline{23} \cup \underline{12} > \underline{13}) &\mapsto \big(\mathbf{U}_{(\gamma,j)}(\underline{23}) \oplus \mathbf{U}_{(\gamma,j)}(\underline{12}) \to \mathbf{U}_{(\gamma,j)}(\underline{13})\big) \\ &= (\mathbf{U}(d_0 d_0(\gamma, j)) \oplus \mathbf{U}(d_0 d_2(\gamma, j)) \to \mathbf{U}(d_0 d_1(\gamma, j))) \end{aligned}$$

remains to be specified. This is determined by $\mathfrak{N}$ by setting

$$\mathbf{U}_{(\gamma,j)}|\mathrm{N}_1(P_{1,3}^{\mathrm{op}}) \coloneqq \mathrm{Op}\mathfrak{N}(\gamma, j). \tag{5.2.7}$$

This is well-defined since $\mathfrak{N}$ sends exit k-paths to $(k-1)$-paths in $B\mathrm{O}$.

Remark 5.2.8 (interrupting the proof)**.** We should note that it is immaterial that (5.2.7) is 'not functorial' (although $\mathbf{U}$ will be). As noted in Lemma 5.2.15, the direct sums appearing in the $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$-face are trivial in that all summands but one are zero, the non-zero one being determined by the exit index j. We use $\mathrm{Op}\mathfrak{N}$ to supply *only the path* in $B\mathrm{O}(m)$. We have $\mathbf{U}_{(\gamma,j)}(\underline{23}) = 0$ if $j = 1$, and $\mathbf{U}_{(\gamma,j)}(\underline{12}) = 0$ if $j = 2$. If $j = 1$, the edges of $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$ are specified by simpliciality as in

$$\begin{tikzcd} & 2 \arrow[dr, "0"] & \\ 1 \arrow[ur, "W"] \arrow[rr, "W"'] & & 3 \end{tikzcd},$$

and $\mathrm{Op}\mathfrak{N}(\gamma, 1)$ is $\mathrm{Op}(\mathrm{id}_W) = \mathrm{id}_W \colon W = 0 \oplus W \to W$. Here, (5.2.7) happens to be functorial as $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$ happens to lie in $\mathcal{V}^{\simeq}$. However, if $j = 2$, the filler of

$$\begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"'] & & 3 \end{tikzcd}$$

is supplied by $\mathrm{Op}\mathfrak{N}(\gamma, 2) = \mathrm{Op}(\gamma_W) = \gamma_W^{-1} \colon W' = 0 \oplus W' \to W$. This breaks functoriality in the sense that $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$ is *not* invertible as a morphism in $\mathcal{V}^{\hookrightarrow}$ from 0 to W. In any case, the path is as desired.

1-paths induced by functoriality. Some 1-paths in the image of $\mathbf{U}(\gamma, j)$ are determined by the data provided thus far and by imposing functoriality (cf. Example 5.1.7). Namely, we have the following decompositions:

$$\begin{aligned}
\text{(i)} = (\underline{0123} > \underline{023}) &= (\underline{23} \cup \underline{012} > \underline{23} \cup \underline{02}) \\
&= \mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}] \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P_{2,3}^{\mathrm{op}}) \times \mathrm{N}_1(P_{0,2}^{\mathrm{op}}) \to \mathrm{N}_1(P_{0,3}^{\mathrm{op}})\right), \\
\text{(i)}' = (\underline{0123} > \underline{013}) &= (\underline{123} \cup \underline{01} > \underline{13} \cup \underline{01}) \\
&= [\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}} \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P_{1,3}^{\mathrm{op}}) \times \mathrm{N}_1(P_{0,1}^{\mathrm{op}}) \to \mathrm{N}_1(P_{0,3}^{\mathrm{op}})\right).
\end{aligned}$$

Thus, functoriality imposes

$$\mathbf{U}_{(\gamma,j)}(\text{i}) = \mathrm{id}_{\mathbf{U}_{(\gamma,j)}(\underline{23})} \oplus \mathbf{U}_{(\gamma,j)}(\underline{012} > \underline{02}), \tag{5.2.9}$$

$$\mathbf{U}_{(\gamma,j)}(\text{i})' = \mathbf{U}_{(\gamma,j)}(\underline{123} > \underline{13}) \oplus \mathrm{id}_{\mathbf{U}_{(\gamma,j)}(\underline{01})} \tag{5.2.10}$$

where the first non-constant summand is determined by (5.2.6) and the second by (5.2.7). In particular, this systematises the ad hoc assignments in (5.1.11) and (5.1.12).

Remaining 1-path and 2-paths. It remains to specify $\mathbf{U}(\gamma, j)$ on the 'long path' in $\mathrm{N}_1(P_{0,3}^{\mathrm{op}})$ and on $\mathrm{N}_2(P_{0,3}^{\mathrm{op}})$. In contrast to paths induced by functoriality, $\underline{03}$ is simple, so

$$\text{(ii)} = (\underline{0123} > \underline{03})$$

presents a genuine choice, and was not handled systematically in Section 5.1.1. It is considered most naturally in conjunction with the two non-degenerate elements in $N_2(P_{0,3}^{\mathrm{op}})$ to be mapped, as it is their (necessarily-)common composition:

$$\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}]"'] \arrow[dr, "{[\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}}}"] \arrow[dd, dotted, "\text{(ii)}" description] & \\
\underline{023} \arrow[dr, "(5.2.6)"'] & & \underline{013} \arrow[dl, "(5.2.6)"] \\
& \underline{03} &
\end{tikzcd} \tag{5.2.11}$$

First, note that, regardless of exit index, this square decomposes into two triangles:

$$\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}]"'] \arrow[dr, "{[\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}}}"] & \\
\underline{023} \arrow[dr, "(5.2.6)"'] & & \underline{013} \arrow[ll, "d_2\gamma" description] \arrow[dl, "(5.2.6)"] \\
& \underline{03} &
\end{tikzcd} \tag{5.2.12}$$

For $j = 2$ (using the labels in (5.1.10)), this reads

$$\begin{tikzcd}[column sep=small] & W' \oplus 0 \oplus V = W' \oplus V \arrow[dl, "\mathrm{id}_{W'} \oplus \gamma_V"'] \arrow[dr, "\gamma_W^{-1} \oplus \mathrm{id}_V"] & \\ W' \oplus V' \arrow[dr, "\gamma_{\oplus'}"'] & & W \oplus V \arrow[ll, "\gamma_W \oplus \gamma_V" description] \arrow[dl, "\gamma_\oplus"] \\ & K & \end{tikzcd},$$

and for $j = 1$ (using the labels in (5.1.13)),

$$\begin{tikzcd}[column sep=small] & 0 \oplus W \oplus V = W \oplus V \arrow[dl, "\mathrm{id}_0 \oplus \gamma_\oplus = \gamma_\oplus"'] \arrow[dr, "\mathrm{id}_{0 \oplus W} \oplus \mathrm{id}_V = \mathrm{id}_W \oplus \mathrm{id}_V"] & \\ 0 \oplus K = K \arrow[dr, "\gamma_K"'] & & W \oplus V \arrow[ll, "\gamma_\oplus" description] \arrow[dl, "\gamma_{\oplus'}"] \\ & K' & \end{tikzcd}.$$

For both indices, the bottom triangle is filled by γ itself, and the top one has a canonical filler. This suggests assigning to (ii) the outer-left concatenation:

$$\mathbf{U}_{(\gamma,j)}(\text{ii}) = \mathbf{U}_{(\gamma,j)}(\underline{0123} > \underline{023}) * \mathbf{U}_{(\gamma,j)}(\underline{023} > \underline{03}). \tag{5.2.13}$$

Accordingly, $\mathbf{U}(\gamma, j)|\mathrm{N}_2(P_{0,3}^{\mathrm{op}})$ is determined by said fillers.

Let us specify the fillers in the case $j = 2$ explicitly.[3] There is an intermediate triangle

$$\begin{tikzcd} W' \oplus V \arrow[d, "\mathrm{id} \oplus \gamma_V"'] \arrow[dr, "\gamma_W^{-1} \oplus \mathrm{id}"] & \\ W' \oplus V & W \oplus V \arrow[l, "\gamma_W \oplus \gamma_V"] \end{tikzcd} \tag{5.2.14}$$

filled by the direct sum of the degenerate 2-path $s_0(\gamma_V)$, i.e.,

$$\begin{tikzcd} V \arrow[d, "\gamma_V"'] \arrow[dr, "\mathrm{id}_V"] & \\ V' & V \arrow[l, "\gamma_V"] \end{tikzcd}$$

and the 2-path

$$\begin{tikzcd} W' \arrow[d, "\mathrm{id}"'] \arrow[dr, "\gamma_W^{-1}"] & \\ W' & W \arrow[l, "\gamma_W"] \end{tikzcd}$$

[3]This will be systematised in Section 5.2.4 and can be skipped. In fact, we will prove that even this concatenation needn't be explicated. The case $j = 2$ is of special importance in said section to the proof of Proposition 5.2.39.

given, writing $\gamma = \gamma_W$ temporarily, by[4]

$$\Gamma\colon \Delta^2 \to BO(m),\ (t_0, t_1, t_2) \mapsto \gamma(t_1, 1 - t_1).$$

Indeed, recalling Notation 5.2.4, we see that its edges are as desired:

$$\begin{aligned}
(d_0\Gamma)(t_0, t_1) &= \Gamma(0, t_0, t_1) = \gamma(t_0, 1 - t_0) = \gamma(t_0, t_1),\\
(d_1\Gamma)(t_0, t_1) &= \Gamma(t_0, 0, t_1) = \gamma(0, 1) = d_1\gamma = W',\\
(d_2\Gamma)(t_0, t_1) &= \Gamma(t_0, t_1, 0) = \gamma(t_1, 1 - t_1) = \gamma(t_1, t_0) = \gamma^{-1}(t_0, t_1).
\end{aligned}$$

The two 2-paths put together (and reparametrised[5]) provide the right half

$$\begin{tikzcd}
 & W' \oplus V \arrow[dl, "\mathrm{id}\oplus\gamma_V"'] \arrow[d, "\mathrm{id}\oplus\gamma_V"] \arrow[dr, "\gamma_W^{-1}\oplus\mathrm{id}"] & \\
W' \oplus V' \arrow[dr, "\gamma_{\oplus'}"'] & W' \oplus V' \arrow[l, "\mathrm{id}"'] \arrow[d, "\gamma_{\oplus'}"] & W \oplus V \arrow[l, "\gamma_W\oplus\gamma_V"'] \arrow[dl, "\gamma_{\oplus}"] \\
 & K &
\end{tikzcd}$$

with the bottom right triangle filled by γ itself. The triangles on the left are both filled by degenerate 2-paths. The (contractible) choice made in pasting is never an issue – we have identified these desired 2-simplices as exactly those that are not required to satisfy any further conditions.

When $j = 1$, the analogous finer triangulation is

$$\begin{tikzcd}
 & W \oplus V \arrow[dl, "\gamma_{\oplus}"'] \arrow[d, "\gamma_{\oplus}"] \arrow[dr, "\mathrm{id}"] & \\
K \arrow[dr, "\gamma_K"'] & K \arrow[l, "\mathrm{id}"'] \arrow[d, "\gamma_K"] & W \oplus V \arrow[l, "\gamma_{\oplus}"'] \arrow[dl, "\gamma_{\oplus'}"] \\
 & K' &
\end{tikzcd}$$

which has the obvious degenerate fillers, and again γ itself in the lower right. This concludes the construction of $\mathbf{U}_{\leq 2}\colon \overline{\mathcal{EX}_{\leq 2}} \to \mathcal{V}^{\hookrightarrow}$.

5.2.3. Ad $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$. Before moving on to the induction step, we will construct $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}\mathbf{U}(\gamma, j)$, the top face, with a closed-form formula (Construction 5.2.21) in every dimension in a way that generalises the above constructions in dimensions ≤ 2.

[4]The point of this elementary exposition is to show that one can fill such diagrams canonically, without having to appeal to non-constructive existence statements, contractibly-unique as the results may be. A systematisation of this construction will play a central role in the proof of Proposition 5.2.39.

[5]One can do this as visually prescribed by the diagram itself. We do not need a general ∞-categorical pasting scheme for this, but can do it instead within $BO(n + m)$. For some related recent progress on this in a slightly different context, see [42] and the references therein.

Lemma 5.2.15. *Let* $(\gamma, j) \in \mathcal{P}_k^\Delta \subset \mathcal{EX}_{k+1}$ *and write, as before,* $\underline{0,i} \mapsto V_{0,i} = (\gamma, j)|_{i-1}$ *for the edges of* $\mathbf{U}(\gamma, j) \in [\mathrm{Path}[k+2], B^\oplus \mathrm{O}]$*, and similarly* $\underline{i,\ell} \mapsto V_{i,\ell}$*, with* $1 \le i < \ell \le k+2$ *throughout.*[6]

(1) *We have* $V_{i,\ell} = \mathfrak{N}(\gamma, j)|_{i-1}$ *for* $1 \le i \le j < \ell \le k+2$*, and zero otherwise.*

(2) *Let* $\alpha_1 < \cdots < \alpha_n$ *be a sequence of natural numbers within the interval* $[1, k+2]$*. Let* $N = N_\alpha \in \{1, \ldots, n\}$ *be the smallest index such that* $\alpha_N > j$ *if it exists, and set* $N = 1$ *otherwise. Then we have*

$$V_{\alpha_1,\ldots,\alpha_n} = \begin{cases} V_{\alpha_{N-1},\alpha_N} = V_{\alpha_{N-1},k+2} \neq 0, & N > 1, \\ 0, & N = 1. \end{cases}$$

Consequently, there are no non-trivial direct sums in the top face.

Proof. First, note that every top edge $V_{\alpha\beta}$, $1 \le \alpha < \beta \le k+2$, is within the fibre $B\mathrm{O}(m)$ of the link projection, since it is the connecting edge in the restriction of $\mathbf{U}(\gamma, j)$ to $\mathrm{Path}[0, \alpha, \beta] \subset \mathrm{Path}[k+2]$, which can be depicted as the 2-face

$$\begin{tikzcd} \alpha \arrow[rr, "V_{\alpha\beta}"] & & \beta \\ & 0 \arrow[ul, "V_{0,\alpha}"] \arrow[ur, "V_{0,\beta}"'] & \end{tikzcd}$$

in $\mathcal{B}^\oplus \mathrm{O}_2 \cong \mathcal{V}_1^{\hookrightarrow}$, given by construction by $(\gamma, j)|_{\alpha-1,\beta-1} \colon V_{0,\alpha} \to V_{0,\beta}$ underlied by $\gamma|_{\alpha-1,\beta-1} \colon V_{\alpha,\beta} \oplus V_{0,\alpha} \to V_{0,\beta}$. Now, by construction, $(\gamma, j)|_{0,\ldots,j-1}$ is low and $(\gamma, j)|_{j,\ldots,n}$ is upper. Thus the restriction of $\mathbf{U}(\gamma, j)$ to $\mathrm{Path}[0, i, \ell]$ for $\ell \le j$ is wholly within $B\mathrm{O}(n)$, so $V_{i,\ell} = 0$. Similarly, it is wholly within $B\mathrm{O}(n+m)$ for $\ell > i > j$, so then $V_{i,\ell} = 0$ as well. Thus, $V_{i,\ell} \neq 0$ implies $1 \le i \le j < \ell \le k+2$.

Conversely, if the inequalities are satisfied, then $V_{0,i-1}$ is low and $V_{0,\ell-1}$ is upper, so the connecting edge $V_{i,\ell} \in B\mathrm{O}(m)$ is non-zero, so in toto the un-equalities specify exactly the non-zero top edges.

Note that

$$V_{i,\ell} = V_{i,\ell'} \text{ if } j < \ell' \le k+2 \text{ as well,} \tag{5.2.16}$$

since $\gamma|_{i-1,\ell-1} \colon V_{i,\ell} \oplus V_{0,i-1} \to V_{0,\ell}$ and $\gamma|_{i-1,\ell'-1} \colon V_{i,\ell'} \oplus V_{0,i-1} \to V_{0,\ell'}$ have the same source $\gamma|_{i-1} = V_{i,\ell} \oplus V_{0,i-1} = V_{i,\ell'} \oplus V_{0,i-1} \in B\mathrm{O}(n+m)_0$. In particular, the first statement is well-defined. That $V_{i,\ell} = \mathfrak{N}(\gamma, j)|_{i-1}$ is immediate: $\mathfrak{N}(\gamma, j) = \mathrm{pr}_2(\gamma|_{1,\ldots,j-1})$, so $\mathfrak{N}(\gamma, j)|_{i-1} = \mathrm{pr}_2(\gamma|_{i-1})$, so $\gamma|_{i,\ell}$ is of type $\gamma|_{i-1,\ell-1} \colon \mathfrak{N}(\gamma, j)|_{i-1} \oplus V_{0,i} \to V_{0,\ell}$.

The second statement is a straightforward consequence. We have

$$V_{\alpha_1,\ldots,\alpha_n} = V_{\alpha_{n-1}\alpha_n} \oplus \cdots \oplus V_{\alpha_{N-1},\alpha_N} \oplus \cdots \oplus V_{\alpha_1,\alpha_2}.$$

By the above, the summands to both sides of $V_{\alpha_{N-1},\alpha_N}$ are zero. That

$$V_{\alpha_{N-1},\alpha_N} = V_{\alpha_{N-1},k+2}$$

[6] We need not assume that $\mathbf{U}$ has been constructed, but may use instead the construction of Section 5.2.1 for these edges, since the statement concerns only the restrictions of $\mathbf{U}(\gamma, j)$ to $\mathrm{Path}[0, \alpha, \beta] \subset \mathrm{Path}[k+2]$.

follows from (5.2.16) by setting $\ell' = k+2$. If there exists no N as described, then every summand is zero since each V_{0,α_k} is in $B\mathrm{O}(n)$. If $N=1$, then each V_{0,α_k} is in $B\mathrm{O}(n+m)$ so that every summand is again zero. □

Remark 5.2.17. The simplification noted in Lemma 5.2.15 is specific to our stratification depth being 1 and not higher. If it was higher, we would see nontrivial sums appearing in the $d_0^{\mathcal{B}^{\oplus}\mathrm{O}}$-face as well.

Before we proceed, let us recall a fundamental fact about the simplicial category $\mathrm{Path}[n]$:

Proposition 5.2.18 ([51, §1.1.5]; [52, 00LL]; [52, 00LM])**.** *For $i, \ell \in [n]$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Path}[n]}(i,\ell) \cong (\Delta[1])^{\times(\ell-i-1)}$$

of simplicial sets. Consequently, there is a canonical homeomorphism

$$\left|\mathrm{Hom}_{\mathrm{Path}[n]}(i,\ell)\right| \cong [0,1]^{\times(\ell-i-1)}.$$

Proof. The first step is the construction of an isomorphism between $(\Delta[1])^{\times n}$ and the nerve of the power poset $\mathbf{P}(\{1,\dots,n\})$ of the set $\{1,\dots,n\}$, ordered by inclusion (and not reverse inclusion). This elementary observation was stated in [52] without proof, so, for completeness, we will provide one.

A vertex of $(\Delta[1])^{\times n}$ is specified exactly by a function from $\{1,\dots,n\}$ to the 2-element set (the vertices of $\Delta[1]$), which specifies exactly a subset of $\{1,\dots,n\}$. More generally, a k-simplex $\phi = (\phi_i)_{i=1}^n$ of $(\Delta[1])^{\times n}$ is a collection of n poset maps $\phi_i\colon [k] \to [1]$; On the other hand, a k-simplex

$$\widehat{\alpha} = \left(\alpha^0 \subseteq \cdots \subseteq \alpha^k\right) \in \mathrm{N}_k\left(\mathbf{P}(\{1,\dots,n\})\right)$$

is a non-decreasing sequence of subsets $\alpha^x \subseteq \{1,\dots,n\}$. Now, we may interpret $\phi_i(x) \in [1]$ as answering the question whether or not the element $i \in \{1,\dots,n\}$ belongs to α^x, the value 0 giving the affirmative. Thus, the function

$$\begin{aligned}(\Delta[1])_k^{\times n} &\to \mathbf{P}(\{1,\dots,n\})_k,\\ \phi = (\phi_i)_{i=1}^n &\mapsto (\alpha_\phi^x)_{x=0}^k\end{aligned}$$

with

$$\alpha_\phi^x = \{i \in \{1,\dots,n\} : \phi_i(x) = 0\}$$

is a bijection. It is well-defined (the subset inclusions hold) since each ϕ_i is a poset map. Varying k, these functions are easily seen to assemble into an isomorphism of simplicial sets.

The subsequent isomorphism

$$\mathrm{N}_\bullet\left(\mathbf{P}(\{1,\dots,n\})\right) \cong \mathrm{N}_\bullet(P_{0,n+1}^{\mathrm{op}})$$

is given by taking complements of subsets and thereby un-un-reversing the order. It is spelled out in [52], as is the homeomorphism that is the second statement. Suffice it to say that a vertex $\beta \in \mathrm{N}_0(P_{0,n+1}^{\mathrm{op}})$ is a subposet of $[n+1]$

of type $\beta = \underline{0, \beta', n+1}$ with $\beta' \subseteq \{1, \ldots, n\}$, and so the rule

$$\begin{aligned} \mathrm{N}_0\left(P^{\mathrm{op}}_{0,n+1}\right) &\to \mathrm{N}_0\left(\mathbf{P}\right) \\ \beta &\mapsto \{1, \ldots, n\} \setminus \beta' \end{aligned}$$

defines a bijection. It is easily seen to lift to an isomorphism of simplicial sets. $\square$

Remark 5.2.19. Because, in the proof of Proposition 5.2.18, the value 0 is (necessarily) taken to give the affirmative, the resulting cubes, when depicted in the standard way (mapping the vertices of $(\Delta[1])^{\times n}$ according to $\phi = (\phi_i)_{i=1}^n \mapsto \sum_i \phi_i(0)\mathbf{i}$ with $\mathbf{i} \in \mathbf{R}^n$ the i'th standard basis vector), will differ from those in Examples 5.2.20 and 5.2.33 below, but only up to a change of basis. For our purposes, this is not a problem: a precise choice of basis will be used only in the proof of Lemma 5.2.37 which is a convexity argument after which the choice may be reverted. Convexity is preserved under linear transformations.

Occasionally, we will call the underlying posets $P^{(\mathrm{op})}_{i,\ell}$ 'cubes' as well. When we do, Proposition 5.2.18 will be understood.

Example 5.2.20. In particular, Lemma 5.2.15 fully specifies the vertices of $P^{\mathrm{op}}_{1,k+2}$ under $\mathbf{U}$. The k-cube $P^{\mathrm{op}}_{1,k+2}$ can be depicted for $k = 3$ as follows:

$$\begin{tikzcd} & \underline{12345} \arrow[rr] \arrow[dl] \arrow[dd] & & \underline{1235} \arrow[dl] \arrow[dd] \\ \underline{1345} \arrow[rr, crossing over] \arrow[dd] & & \underline{135} & \\ & \underline{1245} \arrow[rr] \arrow[dl] & & \underline{125} \arrow[dl] \\ \underline{145} \arrow[rr] & & \underline{15} \arrow[from=uu, crossing over] & \end{tikzcd}$$

- If $j = 1$, the image of this cube under $\mathbf{U}(\gamma, 1)$ must have all vertices equal to $V_{1,5} = \mathfrak{N}_0 := \mathfrak{N}(\gamma, j)|_0$, so non-degenerate sequences $S \in \mathrm{N}_3(P^{\mathrm{op}}_{1,5})$ of arrows from (the image of) $\underline{12345}$ to $\underline{15}$ must all be of type $\mathfrak{N}_0 \to \mathfrak{N}_0 \to \mathfrak{N}_0 \to \mathfrak{N}_0$.
- If $j = 2$, the image under $\mathbf{U}(\gamma, 2)$ must be of type

$$\begin{tikzcd} & \mathfrak{N}_1 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathfrak{N}_1 \arrow[dl] \arrow[dd] \\ \mathfrak{N}_0 \arrow[rr, crossing over] \arrow[dd] & & \mathfrak{N}_0 & \\ & \mathfrak{N}_1 \arrow[rr] \arrow[dl] & & \mathfrak{N}_1 \arrow[dl] \\ \mathfrak{N}_0 \arrow[rr] & & \mathfrak{N}_0 \arrow[from=uu, crossing over] & \end{tikzcd}$$

hence non-degenerate sequences as above, deleting repetitions again, must be of the following types:

$$\mathfrak{N}_1 \to \mathfrak{N}_0 \to \mathfrak{N}_0 \to \mathfrak{N}_0$$
$$\mathfrak{N}_1 \to \mathfrak{N}_1 \to \mathfrak{N}_0 \to \mathfrak{N}_0$$
$$\mathfrak{N}_1 \to \mathfrak{N}_1 \to \mathfrak{N}_1 \to \mathfrak{N}_0$$

On the other hand $\mathrm{Op}\mathfrak{N}(\gamma, j)$ is a $j-1=1$-path of type $(\mathfrak{N}_1 \to \mathfrak{N}_0)$.

- If $j=3$, we have

$$\begin{tikzcd}
& \mathfrak{N}_2 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathfrak{N}_2 \arrow[dl] \arrow[dd] \\
\mathfrak{N}_2 \arrow[rr] \arrow[dd] & & \mathfrak{N}_2 \arrow[dd] & \\
& \mathfrak{N}_1 \arrow[rr] \arrow[dl] & & \mathfrak{N}_1 \arrow[dl] \\
\mathfrak{N}_0 \arrow[rr] & & \mathfrak{N}_0 &
\end{tikzcd}$$

so non-degenerate sequences as above must be of the following types:

$$\mathfrak{N}_2 \to \mathfrak{N}_2 \to \mathfrak{N}_0 \to \mathfrak{N}_0$$
$$\mathfrak{N}_2 \to \mathfrak{N}_2 \to \mathfrak{N}_2 \to \mathfrak{N}_0$$
$$\mathfrak{N}_2 \to \mathfrak{N}_1 \to \mathfrak{N}_0 \to \mathfrak{N}_0$$
$$\mathfrak{N}_2 \to \mathfrak{N}_1 \to \mathfrak{N}_1 \to \mathfrak{N}_0$$
$$\mathfrak{N}_2 \to \mathfrak{N}_2 \to \mathfrak{N}_1 \to \mathfrak{N}_0$$

On the other hand, $\mathrm{Op}\mathfrak{N}(\gamma, j)$ is a $j-1=2$-path of type

$$\begin{tikzcd}
& \mathfrak{N}_1 \arrow[dl] & \\
\mathfrak{N}_0 & & \mathfrak{N}_2 \arrow[ul] \arrow[ll]
\end{tikzcd} .$$

- If $j=4$, we have

$$\begin{tikzcd}
& \mathfrak{N}_3 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathfrak{N}_2 \arrow[dl] \arrow[dd] \\
\mathfrak{N}_3 \arrow[rr] \arrow[dd] & & \mathfrak{N}_2 \arrow[dd] & \\
& \mathfrak{N}_3 \arrow[rr] \arrow[dl] & & \mathfrak{N}_1 \arrow[dl] \\
\mathfrak{N}_3 \arrow[rr] & & \mathfrak{N}_0 &
\end{tikzcd}$$

so non-degenerate sequences as above must be of the following types:

$$\mathfrak{N}_3 \to \mathfrak{N}_3 \to \mathfrak{N}_3 \to \mathfrak{N}_0$$
$$\mathfrak{N}_3 \to \mathfrak{N}_3 \to \mathfrak{N}_2 \to \mathfrak{N}_0$$
$$\mathfrak{N}_3 \to \mathfrak{N}_3 \to \mathfrak{N}_1 \to \mathfrak{N}_0$$
$$\mathfrak{N}_3 \to \mathfrak{N}_2 \to \mathfrak{N}_2 \to \mathfrak{N}_0$$
$$\mathfrak{N}_3 \to \mathfrak{N}_2 \to \mathfrak{N}_1 \to \mathfrak{N}_0$$

We leave a depiction of the $j-1=3$-path $\mathrm{Op}\mathfrak{N}(\gamma, j)$ to the reader.

An examination of Example 5.2.20 is sufficient to reach the following Ansatz:

Construction 5.2.21. Let $(\gamma, j) \in \mathcal{P}_k^{\Delta}$ and let $n \geq 0$ be a natural number. With each sequence $\alpha = (\alpha^0 > \cdots > \alpha^n) \in \mathrm{N}_n(P_{i,j}^{\mathrm{op}})$ where $1 \leq i \leq j < \ell \leq k+2$, we can associate the map

$$\begin{aligned} A\colon [n] &\to [j-1], \\ t &\mapsto j-1-\big(\alpha^t_{N_t-1}-1\big) \end{aligned}$$

in $\mathbf{\Delta}$, where

$$N_t := N_{\alpha^t}$$

is defined with respect to $\alpha^t = (\alpha^t_1 < \cdots < \alpha^t_{n_t})$ as in Lemma 5.2.15. Since $\mathrm{Op}\mathfrak{N}(\gamma, j) \in B\mathrm{O}(m)_{j-1}$, we may pull it back along A for any $\alpha \in \mathrm{N}_n = \mathrm{N}_n(P_{i,j}^{\mathrm{op}})$ to obtain, with a slight abuse of notation, a map

$$\begin{aligned} \mathrm{Op}\mathfrak{N}(\gamma, j)\colon \mathrm{N}_n &\to B\mathrm{O}(m)_n, \\ \alpha &\mapsto A^*\mathrm{Op}\mathfrak{N}(\gamma, j). \end{aligned}$$

Lemma 5.2.22. *For $1 \leq i \leq j < \ell \leq k+2$, the map $\mathrm{Op}\mathfrak{N}(\gamma, j)\colon \mathrm{N}_\bullet(P_{i,\ell}^{\mathrm{op}}) \to B\mathrm{O}(m)_\bullet$ of Construction 5.2.21 is an ∞-functor.*

Proof. We must first prove that the map $A\colon [n] \to [j-1]$ of Construction 5.2.21 is well-defined and monotone. Since N_t is the smallest index such that $\alpha^t_{N_t} > j$ (if it exists), we have $\alpha^t_{N_t-1} \leq j$, and so $A(t) = j - \alpha^t_{N_t-1} \geq 0$. Moreover, $\alpha^t_{N_t-1} \geq 1$ since α^t is a sequence of numbers that starts at $i \geq 1$, hence $A(t) \leq j-1$. We can never have $N_t = 1$ since $i \leq j < \ell$. As for monotonicity, we must show $\alpha^t_{N_t-1} \geq \alpha^{t'}_{N_{t'}-1}$ for $t \leq t'$ in $[n]$. That $\alpha^t > \alpha^{t'}$ means $\alpha^{t'} \subseteq \alpha^t$ as posets. We have $\alpha^t_{N_t-1} = \max\{x \in \alpha^t : x \leq j\}$ and since $\{x \in \alpha^{t'} : x \leq j\} \subseteq \{x \in \alpha^t : x \leq j\}$, we obtain

$$\alpha^{t'}_{N_{t'}-1} = \max\{x \in \alpha^{t'} : x \leq j\} \leq \max\{x \in \alpha^t : x \leq j\} = \alpha^t_{N_t-1}.$$

Incidentally, this also shows $N_t \leq N_{t'}$: there can only be more indices $*$ in $\{1, \ldots, n_{t'}\}$ where $\alpha^{t'}_*$ is at most j, not fewer.

Finally, observe that the map $\phi\colon \mathrm{N}_n \to \mathrm{Hom}_{\mathbf{\Delta}}([n], [j-1])$, $\alpha \mapsto \phi(\alpha) := A$ is manifestly simplicial, and therefore so is $\mathrm{Op}\mathfrak{N}(\gamma, j)$. Indeed, let $\delta\colon [n'] \to [n]$ be a poset map. Then $(\delta^*\alpha)^t = \alpha^{\delta(t)}$ for $t \in [n']$, so

$$\phi(\delta^*\alpha)(t) = (\delta^*\alpha)^t_{N_{(\delta^*\alpha)^t}-1} - 1 = \alpha^{\delta(t)}_{N_{\alpha^{\delta(t)}}-1} - 1 = A(\delta(t)) = \delta^*(\phi(\alpha))(t).$$

□

Remark 5.2.23. Construction 5.2.21 immediately gives

$$\mathrm{Op}\mathfrak{N}(\gamma, j)|_t = \mathfrak{N}(\gamma, j)|_{\alpha^t_{N_t-1}-1},$$

since (cf. [52, 003M]) Op reverses the operation $j-1-$. Consequently,

$$V_\alpha = \mathrm{Op}\mathfrak{N}(\gamma, j)|_\alpha$$

for $\alpha = (\alpha_1 < \cdots < \alpha_n)$ in the situation of Lemma 5.2.15.

Lemma 5.2.24. *The maps of Lemma 5.2.22 lift to a function*

$$\mathrm{Op}\mathfrak{N}\colon \mathcal{P}^\Delta_k \to \left[\mathrm{Path}[1, \ldots, k+2], B^\oplus \mathrm{O}\right].$$

Proof. For pairs $i, \ell \in \{1, \ldots, k+2\}$ that do *not* satisfy $1 \le i \le j < \ell \le k+2$, we let $\mathrm{Op}\mathfrak{N}(\gamma, j)\colon \mathrm{N}_\bullet(P^{\mathrm{op}}_{i,\ell}) \to B\mathrm{O}_\bullet$ (recall Notation 5.1.5) be the constant map to the zero vector space. This defines $\mathrm{Op}\mathfrak{N}(\gamma, j)$ on all morphism spaces, so it remains to verify functoriality, which holds for trivial reasons: if $\alpha\colon i \to \ell$ and $\beta\colon \ell \to \ell'$, then either

- $\mathrm{Op}\mathfrak{N}(\gamma, j)(\alpha) \neq 0$, in which case $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta) = 0$ since $\ell, \ell' > j$, hence $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta \cup \alpha) = 0 \oplus \mathrm{Op}\mathfrak{N}(\gamma, j)(\alpha) = \mathrm{Op}\mathfrak{N}(\gamma, j)(\alpha)$, which holds since $N_{\alpha^*} = N_{(\beta\cup\alpha)^*}$ follows immediately from the definition. Appending β to the head of α does not change the first index for which the sequence becomes larger than j;
- or $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta) \neq 0$ in which case $\mathrm{Op}\mathfrak{N}(\gamma, j)(\alpha) = 0$ since $\ell \le j$, hence $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta \cup \alpha) = \mathrm{Op}\mathfrak{N}(\gamma, j)(\beta) \oplus 0 = \mathrm{Op}\mathfrak{N}(\gamma, j)(\beta)$, which holds, not because $N_{\alpha^*} = N_{(\beta\cup\alpha)^*}$ which is *not* the case here, but because $\beta_{N_\beta - 1} = (\beta \cup \alpha)_{N_{\beta\cup\alpha}-1}$. Appending α to the foot of β does not change the last element in the sequence before it grows larger than j, since that element is already within β;
- or both $\mathrm{Op}\mathfrak{N}(\gamma, j)(\alpha)$ and $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta)$ are zero, in which case it will suffice to show that $\mathrm{Op}\mathfrak{N}(\gamma, j)(\beta \cup \alpha) = 0 = 0 \oplus 0$ as well. This is clear if $i, \ell, \ell' \le j$ or if $i, \ell, \ell' > j$. But $i, \ell \le j$ and $\ell, \ell' > j$ cannot coincide, so these are all the cases.

This argument applies mutatis mutandis to unions of chains of posets to show functoriality on higher morphisms. □

Proposition 5.2.25. *The functions*

$$\mathrm{Op}\mathfrak{N}\colon \mathcal{P}^\Delta_* \to \left[\mathrm{Path}[1, \ldots, *+2], B^\oplus \mathrm{O}\right]$$

of Lemma 5.2.24 extend

$$\mathbf{U}_{\le 1}\colon \overline{\mathcal{E}\mathcal{X}_{\le 1}} \to \mathcal{V}^{\hookrightarrow}.$$

Proof. The only overlap in dimensions ≤ 1 is within $\mathcal{P}^\Delta_0 \subset \mathcal{E}\mathcal{X}_1$. Here, the equality of the two maps is trivial, but we include the proof for completeness. For $(\gamma, 1) \in \mathcal{P}^\Delta_0$ and so for $i = 1$ and $\ell = 2$, we have that $\mathbf{U}(\gamma, 1)|_{\mathrm{N}_0(P^{\mathrm{op}}_{1,2})}\colon \underline{12} \mapsto W = \mathfrak{N}(\gamma, j) = \mathrm{pr}_2(\gamma_0)$ is the normal component of $(\gamma, 1)$. On the other hand, taking $\alpha = \underline{12} \in \mathrm{N}_0(P^{\mathrm{op}}_{1,2})$ yields the constant $A = \mathrm{id}\colon [0] \to [0]$ and so $\mathrm{Op}\mathfrak{N}(\gamma, 1)(\underline{12}) = \mathrm{id}^*\mathrm{Op}\mathfrak{N}(\gamma, j) = \mathrm{Op}(\mathrm{pr}_2(\gamma_0)) = \mathrm{pr}_2(\gamma_0)$

since Op is the identity on vertices. (The compatibility of the vertex values with any extension of $\mathbf{U}_{\leq 1}$ in higher dimensions was noted in Remark 5.2.23.)

Now, for $k \geq 1$, let S be a simplicial operator of type $S = \Sigma^*$ for

$$\Sigma\colon [k+1] \to [1]$$

in $\mathbf{\Delta}$. For $(\gamma, 1) \in \mathcal{P}_0^\Delta$ again, assume that $S(\gamma, 1) \in \mathcal{P}_k^\Delta \subset \mathcal{EX}_{k+1}$ is also vertical. This amounts assuming that Σ is surjective. In this situation, $\mathrm{Op}\mathfrak{N}$ and $\mathbf{U}_{\leq 1}$ may be compared, and we must show that

$$\mathbf{U}_{\leq 1}(S(\gamma, 1))|_{\mathrm{Path}[1,\ldots,k+2]} \stackrel{\text{def}}{=} S\mathbf{U}(\gamma, 1)|_{\mathrm{Path}[1,\ldots,k+2]} = \mathrm{Op}\mathfrak{N}(S(\gamma, 1)) \quad (5.2.26)$$

where the LHS is the restriction to $\mathrm{Path}[1, \ldots, k+2] \subset \mathrm{Path}[k+2]$ of

$$S\mathbf{U}(\gamma, 1)\colon \mathrm{Path}[k+2] \to B^{\oplus}\mathrm{O}.$$

We have $\mathfrak{N}(S(\gamma, 1)) = \mathfrak{N}(S\gamma, \sharp^S 1) = \mathrm{pr}_2(S\gamma|_{0,\ldots,\sharp^S 1})$, where $\sharp^S$ applies $\sharp$'s and $\flat$'s to the exit index 1 according to S. The underlying simplex map Σ is determined by a unique index

$$e_\Sigma \in \{1, \ldots, k+1\}$$

such that $\Sigma(x) = 0$ for $x \leq e_\Sigma - 1$ and $\Sigma(x) = 1$ for $x \geq e_\Sigma$. It is straightforward to see that

$$\sharp^S 1 = e_\Sigma,$$

so

$$\mathrm{Op}\mathfrak{N}(S(\gamma, 1)) = \mathrm{Op}\,\mathrm{pr}_2(S\gamma|_{0,\ldots,e_\Sigma - 1}) = \mathrm{pr}_2(\mathrm{Op}(S\gamma|_{0,\ldots,e_\Sigma - 1})) \in B\mathrm{O}(m)_{e_\Sigma - 1}.$$

But

$$\begin{aligned}(S\gamma)|_{0,\ldots,e_\Sigma - 1} &= ([e_\Sigma - 1] \hookrightarrow [k+1] \xrightarrow{\Sigma} [1])^*(\gamma) \\ &= ([e_\Sigma - 1] \to [0] \hookrightarrow [1])^*\gamma \\ &= (s_0)^{e_\Sigma - 1} d_1 \gamma \\ &= (s_0)^{e_\Sigma - 1} \gamma_0,\end{aligned}$$

and $\mathrm{Op}s_0 = s_{0-0}\mathrm{Op} = s_0\mathrm{Op}$ in dimension 0, and pr_2 commutes with Op as well as with any simplicial operator, so we have

$$\begin{aligned}\mathrm{Op}\mathfrak{N}(S(\gamma, 1))\colon \alpha \mapsto & A^*\mathrm{Op}\,\mathrm{pr}_2(S\gamma|_{0,\ldots,e_\Sigma - 1}) = \mathrm{pr}_2 A^* \mathrm{Op}(s_0)^{e_\Sigma - 1}\gamma_0 \\ &= \mathrm{pr}_2 A^*(s_0)^{e_\Sigma - 1}\mathrm{Op}\gamma_0 = \mathrm{pr}_2 A^*(s_0)^{e_\Sigma - 1}\gamma_0 \\ &= \mathrm{pr}_2(s_0)^n\gamma_0 = (s_0)^n \mathrm{pr}_2\gamma_0\end{aligned}$$

for $\alpha \in \mathrm{N}_n(P_{i,\ell}^{\mathrm{op}})$ and $1 \leq i \leq e_\Sigma < \ell \leq k+2$, and (the n-fold degenerate) zero otherwise.

On the other hand, $\Sigma\colon [k+1] \to [1]$ induces a map

$$\Sigma_{+1}\colon [k+2] \to [2]$$

defined by

$$\Sigma_{+1}(0) = 0, \quad \text{and} \quad \Sigma_{+1}(i) = \Sigma(i-1) + 1$$

for $1 \le i \le k{+}2$. Along the under-∞-category identifications $\mathcal{V}^{\hookrightarrow}_{*} \cong (\mathcal{B}^{\oplus}\mathrm{O})_{*+1}$, we have that $S(\mathbf{U}(\gamma,1)) \in \mathcal{V}^{\hookrightarrow}_{k+1}$ corresponds to

$$S_{+1}(\mathbf{U}(\gamma,1)) \coloneqq \Sigma^{*}_{+1}(\mathbf{U}(\gamma,1))$$

in $(\mathcal{B}^{\oplus}\mathrm{O})_{k+2}$ (see Lemma 4.3.4), where we identified $\mathbf{U}(\gamma,1)$ with the corresponding element

$$\mathbf{U}(\gamma,1)\colon \mathrm{Path}[2] \to B^{\oplus}\mathrm{O} \quad \text{in} \quad (\mathcal{B}^{\oplus}\mathrm{O})_2.$$

The restriction of the induced map $\Sigma_{+1}\colon \mathrm{Path}[k{+}2] \to \mathrm{Path}[2]$ to $\mathrm{Path}[1,\dots,k{+}1]$ factors through $\mathrm{Path}[1,2] \subset \mathrm{Path}[2]$ by construction and defines the LHS of (5.2.26) as the composition

$$\mathrm{Path}[1,\dots,k+2] \xrightarrow{\Sigma_{+1}|} \mathrm{Path}[1,2] \xrightarrow{\mathbf{U}(\gamma,1)|} B^{\oplus}\mathrm{O}.$$

Let now e_Σ be as above, so that $e_\Sigma + 1 \in \{2,\dots,k+2\}$ fulfills the analogous function for the restriction $\Sigma_{+1}|$: we have $\Sigma_{+1}(x) = 1$ for $1 \le x \le e_\Sigma = e_\Sigma + 1 - 1$ and $\Sigma(x) = 2$ for $x \ge e_\Sigma + 1$. Suppose, then, that $1 \le i < \ell \le k+2$. We have $S\mathbf{U}(\gamma,1)\colon \underline{i,\ell} \mapsto \mathbf{U}(\gamma,1)(\mathrm{id}_1) = \mathrm{id}_{B^{\oplus}\mathrm{O}} = 0 \in B\mathrm{O}$ if $\Sigma_{+1}(i) = \Sigma_{+1}(\ell) = 1$, i.e., if $i,\ell \le e_\Sigma$. Similarly, $S\mathbf{U}(\gamma,1)\colon \underline{i,\ell} \mapsto \mathbf{U}(\gamma,1)(\mathrm{id}_2) = 0$ if $\Sigma_{+1}(i) = \Sigma_{+1}(\ell) = 2$, i.e., if $i,\ell > e_\Sigma$. If, however, $i \le e_\Sigma$ and $\ell > e_\Sigma$, then $S\mathbf{U}(\gamma,1)\colon \underline{i,\ell} \mapsto \mathbf{U}(\gamma,1)(\underline{12}) = W = \mathrm{pr}_2(\gamma_0)$. In both cases, the result coincides with the value of the RHS of (5.2.26).

Generalising this observation to higher dimensions is straightforward: let $\alpha \in \mathrm{N}_n(P^{\mathrm{op}}_{i,\ell})$. If $i,\ell \le e_\Sigma$ or $i,\ell > e_\Sigma$, then $S\mathbf{U}(\gamma,1)\colon \alpha \mapsto (s_0)^n 0$, and if $i \le e_\Sigma$ and $\ell > e_\Sigma$, then $S\mathbf{U}(\gamma,1)\colon \alpha \mapsto (s_0)^n \mathrm{pr}_2\gamma_0$. $\square$

5.2.4. The induction step. Let us write

$$\overline{\mathcal{E}\mathcal{X}_{\le k+1}} \coloneqq \overline{(\mathcal{E}\mathcal{X}_{\le k+1} \smallsetminus \overline{\mathcal{E}\mathcal{X}_{\le k}}) \cup \overline{\mathcal{E}\mathcal{X}_{\le k}}}$$

for the simplicial subset of $\mathcal{E}\mathcal{X}$ generated by $\mathcal{E}\mathcal{X}_{\le k}$ together with the non-degenerate exit $(k+1)$-paths in $\mathcal{P}^{\Delta}_k \subset \mathcal{E}\mathcal{X}_{k+1}$ (cf. Remark 5.1.18), and let us assume a map

$$\mathbf{U}_{\le k}\colon \overline{\mathcal{E}\mathcal{X}_{\le k}} \to \mathcal{V}^{\hookrightarrow}$$

is given which satisfies

$$\mathbf{U}_{\le k}|_{\le 1} = \mathbf{U}_{\le 1} \tag{5.2.27}$$

and

$$d_0^{\mathcal{B}^{\oplus}\mathrm{O}}\mathbf{U}_{\le k}(\gamma,j) \overset{\mathrm{def}}{=} \mathbf{U}_{\le k}(\gamma,j)|_{\mathrm{Path}[1,\dots,*+2]} = \mathrm{Op}\mathfrak{N}(\gamma,j) \tag{5.2.28}$$

for all $(\gamma,j) \in \mathcal{P}^{\Delta}_{*} \subset (\overline{\mathcal{E}\mathcal{X}_{\le k+1}})_{*+1}$, with $\mathrm{Op}\mathfrak{N}$ defined as in Construction 5.2.21. Proposition 5.2.25 states exactly that this equality holds in the base case $k = 1$. Moreover, this is consistent by Lemma 5.2.15: as noted there, the condition that $\mathbf{U}_{\le k}$ extend $\mathbf{U}_{\le 1}$ fixes the spaces $V_{i,\ell}$ for $1 \le i < \ell \le k+2$ through the induced restrictions to the subcategories $\mathrm{Path}[0,\alpha,\beta] \subset \mathrm{Path}[k{+}2]$, and these spaces coincide with the values of $\mathrm{Op}\mathfrak{N}$ wherever they overlap.

There is a final inductive assumption which we will formulate and justify in the proof of Proposition 5.2.39; see immediately after (5.2.49). It involves a construction that becomes necessary for the first time within said proof, which is why we chose to formulate it therein.

Now, having fixed $\mathbf{U}_{\leq k}(\gamma, j)$ on every $\mathcal{B}^{\oplus}$O-face, i.e., on every proper simplicial subcategory of Path$[k+2]$, we have in particular fixed it on every hom-space $\mathrm{Hom}_{\mathrm{Path}[k+2]}(i,\ell)$ for $(i,\ell) \neq (0, k+2)$.

It remains then to specify it on all of $\mathrm{Hom}_{\mathrm{Path}[k+2]}(0, k+2) = \mathrm{N}_{\bullet} P^{\mathrm{op}}_{0,k+2}$. The initial object of $P^{\mathrm{op}}_{0,k+2}$ is $[k+2] = \underline{0, \dots, k+2}$, and its final object is $\underline{0, k+2}$. All arrows with domain $[k+2]$ are composite (recall Definition 5.1.6) *except* for the arrow $[k+2] > \underline{0, k+2}$, so this is the only 1-morphism of Path$[k+2]$ whose image is not determined by the inductive hypothesis. For instance, $\underline{01234} > \underline{014} = (\underline{01} > \underline{01}) \cup (\underline{1234} > \underline{14})$, and the value of both factors is determined already by $\mathbf{U}_{\leq k}$. Generally, any 1-morphism $[k+2] > [k+2] \smallsetminus \beta$ with $\beta \subset \{1, \dots, k+1\}$ a proper subset is determined by functoriality in the same way, as can be seen by decomposing the target nontrivially and using the initiality of both factors in the corresponding decomposition of $[k+2]$. Namely, take $\delta \in \{1, \dots, k+1\} \smallsetminus \beta$, and write

$$[k+2] \smallsetminus \beta = (\underline{0, \dots, \delta} \cap ([k+2] \smallsetminus \beta)) \cup (\underline{\delta, \dots, k+2} \cap ([k+2] \smallsetminus \beta)) = \beta^{\mathrm{c}}_{L} \cup \beta^{\mathrm{c}}_{R}.$$

Then

$$[k+2] \smallsetminus \beta > \underline{0, k+2} = (\underline{0, \dots, \delta} > \beta^{\mathrm{c}}_{L}) \cup (\underline{\delta, \dots, k+2} > \beta^{\mathrm{c}}_{R}).$$

Consequently, the images of all higher morphisms of whom $[k+2] > \underline{0, k+2}$ is a side are likewise undetermined; since we are mapping from the nerve of $P^{\mathrm{op}}_{0,k+2}$, this means (the images of the) sequences with long edge $[k+2] > \underline{0, k+2}$. By the *long edge* of an n-chain $S \in \mathrm{N}_n$, $n \geq 1$, we mean its pullback along $[1] \hookrightarrow [n]$, $0 \mapsto 0$; $1 \mapsto n$. Clearly, if $[k+2] > \underline{0, k+2}$ is an edge of S, then it is also its long edge since $\underline{0, k+2}$ is final and $[k+2]$ is initial.

It will therefore suffice to provide construct images for the simplices of $\mathrm{N}_{\bullet}(P^{\mathrm{op}}_{0,k+2})$ with long edge $[k+2] > \underline{0, k+2}$, as well as an image for this 1-morphism itself. We have seen that the image of the latter need obey no other condition. Our strategy will be to generalise the idea of the decomposition in (5.2.12). To this end, we will first identify where exactly the path $\gamma \in B\mathrm{O}(n+m)_{k+1}$ fits in the image of the $(k+1)$-cube $P_{0,k+2}$.

Let us write $\mathfrak{N}_* \coloneqq \mathfrak{N}(\gamma, j)|_*$, as in Example 5.2.20.

Lemma 5.2.29. *In the situation of Lemma 5.2.15, let $\alpha = (\alpha_1 < \cdots < \alpha_n)$ be a sequence within $[1, k+2]$. Then we have*

$$\mathbf{U}_{(\gamma,j)}(\underline{0, \alpha}) = \begin{cases} \mathfrak{N}_{\alpha_{N-1}-1} \oplus (\gamma, j)|_{\alpha_1 - 1}, & N > 1, \\ (\gamma, j)|_{\alpha_1 - 1}, & N = 1. \end{cases}$$

Proof. This follows immediately from Lemma 5.2.15 and Proposition 5.2.25 after decomposing the space as $V_{0,\alpha} = V_{\alpha} \oplus V_{0,\alpha_1}$. □

Corollary 5.2.30. *In the situation of Lemma 5.2.15, for $0 < i < k+2$, we have*

$$\mathbf{U}_{(\gamma,j)}(\underline{0, i, k+2}) = \gamma|_{i-1},$$

and

$$\mathbf{U}_{(\gamma,j)}(\underline{0, k+2}) = \gamma|_{k+1}.$$

Proof. The second equality is clear. Now, Lemma 5.2.29 yields

$$\mathbf{U}_{(\gamma,j)}(\underline{0,i,k+2}) = \begin{cases} \mathfrak{N}_{i-1} \oplus (\gamma,j)|_{i-1}, & i \leq j \\ (\gamma,j)|_{i-1}, & i > j. \end{cases}$$

If $i \leq j$, then $(\gamma,j)|_{i-1}$ is low, so $\gamma|_{i-1} = \mathfrak{N}_{i-1} \oplus \pi\,(\gamma|_{i-1})$, and if $i > j$, then $(\gamma,j)|_{i-1}$ is upper, so $\gamma|_{i-1} = (\gamma,j)|_{i-1}$ (recall that we suppress ι). $\square$

Now, all of $\mathrm{N}_\bullet(P^{\mathrm{op}}_{0,k+2})$ is to map to $B\mathrm{O}(n+m) = \mathrm{Sing}_\bullet B\mathrm{O}(n+m)$ under $\mathbf{U}(\gamma,j)$. Therefore, by the adjunction between $|-|$ and $\mathrm{Sing}_\bullet$, this is equivalent to mapping out of $|\mathrm{N}_\bullet|$ to the space $B\mathrm{O}(n+m)$ instead. We will specify the 'location' of γ inside (the image of) $\mathrm{N}_\bullet$ by means of an embedding $\Delta^{k+1} \hookrightarrow |\mathrm{N}_\bullet|$.

Construction 5.2.31. The vertices featuring in Corollary 5.2.30 specify a (topological) $(k+1)$-simplex within $|\mathrm{N}_\bullet|$. Namely, since, by Proposition 5.2.18, $|\mathrm{N}_\bullet|$ is canonically homeomorphic to the $(k+1)$-cube, we may define

$$\triangledown \colon \Delta^{k+1} \hookrightarrow |\mathrm{N}_\bullet|$$

to be (i.e., map homeomorphically onto) the subset of $|\mathrm{N}_\bullet|$ given by the convex hull of the vertices $\{\underline{0,i,k+2} : 0 < i < k+2\} \cup \{\underline{0,k+2}\} \subset |\mathrm{N}_\bullet| \cong [0,1]^{\times(k+1)}$. This is indeed a topological $(k+1)$-simplex: shifting and rotating the cube so that the (image under this homeomorphism of) $\underline{0,k+2}$ lies at the origin, it is easily checked (vis-à-vis the proof of Proposition 5.2.18, keeping in mind Remark 5.2.19) that the points (given by the images of) $\underline{0,i,k+2}$ give exactly the unit vectors on the coordinate axes.

Writing

$$\vartriangle := \overline{|\mathrm{N}_\bullet| \smallsetminus \triangledown}$$

for the closure within $[0,1]^{\times(k+1)}$ of the complement, we obtain the decomposition

$$|\mathrm{N}_\bullet| \cong \vartriangle \cup_\partial \triangledown \tag{5.2.32}$$

where the common boundary ∂ is the convex hull of the vertices $\{\underline{0,i,k+2} : 0 < i < k+2\}$.

Example 5.2.33. Consider an exit 3-path $(\gamma,3) \in \mathcal{P}_2^\Delta$ of index 3:

$$\begin{tikzcd} & & K \\ & W_1 \oplus V_1 \arrow[ur, dotted] \arrow[dr, dotted] & \\ W_0 \oplus V_0 \arrow[rr] \arrow[ur, dotted] \arrow[uurr, bend left] & & W_2 \oplus V_2 \arrow[uu] \end{tikzcd}.$$

The diagram depicts the underlying path $\gamma \in B\mathrm{O}(n+m)_3$. The edges of $\mathbf{U}(\gamma,3) \in \mathcal{B}^\oplus\mathrm{O}_4$ are given, due to (5.2.27), as follows:

$$\begin{cases} \underline{01} \mapsto V_0,\ \underline{02} \mapsto V_1,\ \underline{03} \mapsto V_2,\ \underline{04} \mapsto K, \\ \underline{14} \mapsto W_0,\ \underline{24} \mapsto W_1,\ \underline{34} \mapsto W_2 \end{cases}$$

and the remaining edges are zero. Now, the image of the 3-cube $P^{\mathrm{op}}_{0,4}$

$$\begin{tikzcd}
& \underline{01234} \arrow[rr] \arrow[dl] \arrow[dd] & & \underline{0124} \arrow[dl] \arrow[dd] \\
\underline{0234} \arrow[rr] \arrow[dd] & & \underline{024} \arrow[dd] & \\
& \underline{0134} \arrow[rr] \arrow[dl] & & \underline{014} \arrow[dl] \\
\underline{034} \arrow[rr] & & \underline{04} &
\end{tikzcd}$$

under $\mathbf{U}(\gamma, j)$ looks as follows:

$$\begin{tikzcd}
& W_2 \oplus V_0 \arrow[rr, "\gamma^{-1}_{W12}\oplus\mathrm{id}"] \arrow[dl, "\mathrm{id}\oplus\gamma_{V01}"'] \arrow[dd, equal] & & W_1 \oplus V_0 \arrow[dl, "\mathrm{id}\oplus\gamma_{V01}"'] \arrow[dd, "\gamma^{-1}_{W01}\oplus\mathrm{id}"] \\
W_2 \oplus V_1 \arrow[rr, "\mathrm{id}^{-1}_{W12}\oplus\mathrm{id}"] \arrow[dd, "\mathrm{id}\oplus\gamma_{V12}"'] & & W_1 \oplus V_1 \arrow[dd, "\mathrm{id}_{\oplus 1}"] & \\
& W_2 \oplus V_0 \arrow[rr, "\gamma^{-1}_{W02}\oplus\mathrm{id}"] \arrow[dl, "\mathrm{id}\oplus\gamma_{V02}"'] & & W_0 \oplus V_0 \arrow[dl, "\gamma_{\oplus 0}"] \\
W_2 \oplus V_2 \arrow[rr, "\gamma_{\oplus 2}"] & & K &
\end{tikzcd}$$

Painting in the outer-left concatenation chosen to be the image of $\underline{01234} > \underline{04}$ according to the discussion in Section 5.2.2 (in green) and the (edges of the) 'lower' tetrahedron given by $\gamma \in BO_3$ itself (in blue), we see that

$$\begin{tikzcd}
& W_2 \oplus V_0 \arrow[rr] \arrow[dl, green] \arrow[dd, equal] & & W_1 \oplus V_0 \arrow[dl] \arrow[dd] \\
W_2 \oplus V_1 \arrow[rr] \arrow[dd, green] & & W_1 \oplus V_1 \arrow[dd, blue] \arrow[dll, blue, "\gamma_{W12}\oplus\gamma_{V12}"'] & \\
& W_2 \oplus V_0 \arrow[rr] \arrow[dl] & & W_0 \oplus V_0, \arrow[dl, blue] \arrow[ul, blue, "\gamma_{W01}\oplus\gamma_{V01}"'] \arrow[llld, blue, "\gamma_{W02}\oplus\gamma_{V02}"] \\
W_2 \oplus V_2 \arrow[rr, cyan] & & K &
\end{tikzcd}$$

homotopy-commutes by inspection. In terms of Construction 5.2.31, the 3-simplex with the blue/cyan edges geometrically-realises to $\bigtriangledown$, and the cube with $\bigtriangledown$ cut off gives $\bigtriangleup$. The 2-dimensional instantiation of this idea is depicted in (5.2.12).

As Example 5.2.33 suggests, the next step in our strategy is to define a new poset $\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}$ such that the full subposet $P^{\mathrm{op}}_{0,k+2} \setminus \{0, \underline{k+2}\}$ of the original poset given by removing its final object is embedded into it,

$$P^{\mathrm{op}}_{0,k+2} \smallsetminus \{\underline{0,k+2}\} \subset \overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}, \tag{5.2.34}$$

and such that we have a homeomorphism

$$\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right| \cong \triangle. \tag{5.2.35}$$

Definition 5.2.36. By $\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}$ we denote the poset whose objects are the same as those of $P^{\mathrm{op}}_{0,k+2} \smallsetminus \{\underline{0,k+2}\}$, and whose arrows are those of the latter together with the new primitive arrows

$$\underline{0,i,k+2} \to \underline{0,\ell,k+2}$$

whenever $0 < i < \ell < k+2$.

Lemma 5.2.37. *Definition 5.2.36 satisfies* (5.2.34) *and* (5.2.35).

Proof. That (5.2.34) is satisfied is clear by the construction. Let us observe now that (5.2.34) lifts as in the diagram

$$\begin{tikzcd}
\left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+2} \smallsetminus \{\underline{0,k+2}\}\right)\right| \arrow[r, hook] \arrow[d, hook] & \left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right| \arrow[dl, hook, dotted] \\
{[0,1]^{\times(k+1)}} &
\end{tikzcd}$$

to an embedding into the $(k+1)$-cube. Indeed, any topological simplex in $\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right|$ can be sent to the convex hull of the images of its vertices within $[0,1]^{\times(k+1)}$. This makes the diagram above commute, since the map

$$\left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+2} \smallsetminus \{\underline{0,k+2}\}\right)\right| \hookrightarrow [0,1]^{\times(k+1)}$$

is given by restricting $|\mathrm{N}_\bullet(P_{0,k+2})| \hookrightarrow [0,1]^{\times(k+1)}$, and the latter is easily seen (by examining the proof of Proposition 5.2.18) to be itself defined by sending simplices to the convex hulls of the images of their vertices. (The images of the vertices are fixed explicitly in said proof.)

Note now that the intersection of the image $\widetilde{\triangle}$ of $\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right|$ and ∇ is given exactly by the convex hull of $\{\underline{0,i,k+2} : 0 < i < k+2\}$, which is by construction the image of the geometric realisation of the k-simplex

$$(\underline{0,1,k+2} \to \underline{0,2,k+2} \to \cdots \to \underline{0,k+1,k+2})$$

from $\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)$. Therefore, we can glue with ∇ and stay within the cube:

$$\widetilde{\triangle} \cup_\partial \nabla \subseteq [0,1]^{\times(k+1)}.$$

It remains to show the reverse inclusion, which will imply that $\widetilde{\triangle}$, and therefore $\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right|$, are homeomorphic to $\triangle$.

The $(k+1)$-cube itself being the convex hull of its corners, it is equal to the convex hull of the union of $\widetilde{\triangle}$ and $\triangledown$. By [66, Theorem 3.3], then,[7] it is the union of all convex combinations of these two sets:

$$[0,1]^{\times(k+1)} = \bigcup_{\substack{\lambda_i \geq 0,\\ \lambda_1+\lambda_2=1}} \left(\lambda_1\widetilde{\triangle} + \lambda_2\triangledown\right)$$

This means that it suffices to show that given points $x \in \widetilde{\triangle}$ and $y \in \triangledown$, the line segment L connecting x and y lies within $\widetilde{\triangle} \cup \triangledown$. This is a straightforward application of the intermediate value theorem, as we will now show.

Shifting and rotating the cube such that the image $\underline{0, k+2}$ is at the origin and the image $\mathbf{i} = (0,\ldots,0,1,0,\ldots,0) \in \mathbf{R}^{k+1}$ of $\underline{0, i, k+2}$ is the i'th unit vector among the $k+1$ coordinate axes (cf. the discussion in Construction 5.2.31), we place L as well as all of the $(k+1)$-cube inside the non-negative orthant of $\mathbf{R}^{n+1}$. We may assume that neither point lies on the common boundary

$$\partial = \left\{ \sum_{i=1}^{k+1} \lambda_i \mathbf{i} : \sum \lambda_i = 1 \right\},$$

since otherwise L lies within (at least) one of the two sets by their convexity, and we are done. Now, $\triangledown$ is the convex hull of $\{0, \mathbf{i}\}_{i=1}^{k+1}$, so

$$y = \sum_{i=1}^{k+1} \lambda_i \mathbf{i} \quad \text{with} \quad \sum \lambda_i < 1.$$

On the other hand, $\widetilde{\triangle}$ is the convex hull of

$$\left\{ \sum_{i=1}^{k+1} \mu_i \mathbf{i} : \mu_i \in \{0,1\},\ (\mu_i)_{i=1}^{k+1} \neq 0^{\times(k+1)} \right\}$$

where there exist evidently $2^{k+1}-1$ possibilities for $\mu = (\mu_i)_{i=1}^{k+1} \in \{0,1\}^{\{1,\ldots,k+1\}} \smallsetminus \{\mathrm{const}_0\}$. We obtain

$$x = \sum_{\substack{j \in \{1,\ldots,2^{k+1}-1\}\\ \sum \lambda_j = 1}} \lambda_j \sum_{\substack{i \in \{1,\ldots,k+1\}\\ \mu_i^j \in \{0,1\}\\ \sum_i \mu_i^j \geq 1}} \mu_i^j \mathbf{i}$$

and, writing $\lambda_i' := \sum_{\substack{j \in \{1,\ldots,2^{k+1}-1\}\\ \sum \lambda_j = 1}} \lambda_j \mu_i^j$, we have

$$x = \sum_{i=1}^{k+1} \lambda_i' \mathbf{i} \quad \text{with} \quad \sum_i \lambda_i' \neq 1$$

[7]It is *not* true in general that the convex hull of a collection of points in euclidean space is the union of the convex hulls of two subsets that partition the collection with non-empty intersection.

due to $x \notin \partial$ by assumption. But since $\sum_i \mu_i^j \geq 1$ and $\sum \lambda_j = 1$, we have $\sum \lambda_i' \geq 1$ in any case, so we must have

$$\sum \lambda_i' > 1$$

Thus, the continuous function

$$L \to \mathbf{R},$$
$$\sum \nu_i \mathbf{i} \mapsto 1 - \sum \nu_i$$

is negative at x and positive at y, and so is zero at some $Z \in L$. But then $Z \in \partial$, so, writing

$$L = L_{yZ} \cup_Z L_{Zx}$$

where $L_{\alpha\beta}$ is the line segment connecting the points α and β, we have, since $Z \in \widetilde{\triangle} \cap \nabla$, that $L_{yZ} \subseteq \nabla$ and $L_{Zx} \subseteq \widetilde{\triangle}$ by convexity, and therefore $L \subseteq \widetilde{\triangle} \cup \nabla$. This implies $[0,1]^{\times(k+1)} \subseteq \widetilde{\triangle} \cup \nabla$, as desired. □

We summarise the resulting strategy of proof for Theorem 5.1.3 in the following

Corollary 5.2.38. *Providing, for each* $(\gamma, j) \in \mathcal{P}_k^{\Delta} \subset \mathcal{EX}_{k+1} \smallsetminus \overline{\mathcal{EX}_{\leq k}}$, *an* ∞*-functor*

$$\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right) \to B\mathrm{O}(n+m),$$

whose

- *restriction to* $\mathrm{N}_\bullet\left(P_{0,k+2}\right) \smallsetminus \{\underline{0, k+2}\}$ *agrees with the restriction of* $\mathbf{U}_{\leq k}$, *and whose*
- *value on* $\left(\underline{0,1,k+2} \to \underline{0,2,k+2} \to \cdots \to \underline{0,k+1,k+2}\right) \in \mathrm{N}_k\left(\overline{P_{0,\widehat{k+2}}}\right)$ *is* $d_{k+1}(\gamma) \in B\mathrm{O}(n+m)_k$

yields an extension $\mathbf{U}_{\leq k+1} \colon \overline{\mathcal{EX}_{k+1}} \to \mathcal{V}^{\hookrightarrow}$ *of* $\mathbf{U}_{\leq k}$.

Proof. First, let us summarise what we have already proved. Given such an exit path (γ, j), it suffices, by the discussion at the beginning of this section, to provide an extension of $\mathbf{U}_{\leq k}$ on $\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+2}\right)$. Using the bijection

$$\mathrm{Hom}_{\mathrm{sSet}}\left(\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+2}\right), \mathrm{Sing}_\bullet B\mathrm{O}(n+m)\right) \cong \mathrm{Hom}_{\mathrm{Top}}\left(\left|\mathrm{N}_\bullet\right|, B\mathrm{O}(n+m)\right),$$

and combining Lemma 5.2.37 with (5.2.32) from Construction 5.2.31, we see that it suffices to provide two maps of type

$$\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+2}}}\right)\right| \to B\mathrm{O}(n+m)$$
$$\nabla \cong \Delta^{k+1} \to B\mathrm{O}(n+m)$$

such that

- they agree on $\partial \cong \Delta^k$, the convex hull of $\{\underline{0,i,k+2} : 0 < i < k+2\}$, which is the intersection within the unit $(k+1)$-cube of the two domains; and
- they extend $\mathbf{U}_{\leq k}$.

Now, we can define ∇ to map to $\gamma \in BO(n+m)_{k+1}$. Since $\nabla \cong \Delta^{k+1}$ is given by identifying $\underline{0,i,k+2}$ with $i-1$ and $\underline{0,k+2}$ with $k+1$, we see that the sequence $\left(\underline{0,1,k+2} \to \cdots \to \underline{0,k+1,k+2}\right) \in \mathrm{N}_k\left(\overline{P_{0,\widehat{k+2}}}\right)$ is exactly the $(k+1)$'st face, so its value is $d_{k+1}\gamma$ by construction.

The statement will follow once we show that $\gamma\colon \nabla \to BO(n+m)$ itself is compatible with $\mathbf{U}_{\leq k}$. The only definitional overlap is at the vertices and those edges that are of type $\underline{0,i,k+2} > \underline{0,k+2}$, since the other edges $\underline{0,i,k+2} \to \underline{0,\ell,k+2}$, $i < \ell$, are not in $P^{\mathrm{op}}_{0,k+2}$. This verifies that there are no overlapping higher simplices. Now, Corollary 5.2.30 states exactly that the values of the vertices agree. The value of $\mathbf{U}_{\leq k}$ on the edges is fixed by $\mathbf{U}_{\leq k}|_{\leq 1} = \mathbf{U}_{\leq 1}$, the inductive hypothesis (5.2.27). Here, we see that there is agreement by construction: setting $\gamma' := \left(\Delta\{i-1,k+1\} \hookrightarrow \Delta[k+1]\right)^{*}\gamma$, we apply the definition from Section 5.2.1: $\mathbf{U}_{\leq 1}(\gamma',1)(\underline{012} > \underline{02}) = \gamma'$. $\square$

Notation. $\triangle = \triangle_{\bullet} := \mathrm{N}_{\bullet}\left(\overline{P_{0,\widehat{k+2}}}\right)$, $\partial := \left(\underline{0,1,k+2} \to \cdots \to \underline{0,k+1,k+2}\right) \in \triangle_k$.

By Corollary 5.2.38, the following concludes the proof of Theorem 5.1.3.

Proposition 5.2.39. *For each $(\gamma,j) \in \mathcal{P}^{\triangle}_k \subset \mathcal{E}\mathcal{X}_{k+1} \smallsetminus \overline{\mathcal{E}\mathcal{X}_{\leq k}}$, there exists an ∞-functor*
$$\mathbf{U}^{\triangle}\colon \triangle \to BO(n+m)$$
that extends $\mathbf{U}_{\leq k}$ and is such that $\mathbf{U}^{\triangle}(\partial) = d_{k+1}(\gamma)$.

Proof. All simplices in $\triangle$ that have at most one vertex from ∂ are already determined by $\mathbf{U}_{\leq k}$ and functoriality, as noted at the beginning of this section. Consequently, the non-degenerate $(k+1)$-simplices of $\triangle$ are exactly those that possess non-degenerate edges in ∂. Setting $\mathbf{U}^{\triangle}(\partial) = d_{k+1}(\gamma)$, we will exhibit natural fillers for these, generalising the observations in Section 5.2.2. Let
$$\underline{0,\alpha^0,k+2} \geq \cdots \geq \underline{0,\alpha^r,k+2} \geq \underline{0,\beta_{r+1},k+2} \to \cdots \to \underline{0,\beta_{r+s},k+2} \tag{5.2.40}$$
be an element of $\triangle_{r+s}$, which (the element) we will denote by X, where, with a slight abuse of notation,
$$[1,k+1] \supseteq \alpha^0 \supseteq \cdots \supseteq \alpha^r \supseteq \beta_{r+1} \leq \cdots \leq \beta_{r+s}$$
with non-empty sequences $\alpha^i = \left(\alpha^i_j\right)_{j=1}^{r_i}$ and elements β_i, $s \geq 1$. Observe that if $N_{\alpha^i} = 1$, then also $N_{\alpha^{i'}} = 1$ for all $i' \geq i$. We have $\alpha^0_1 \leq \cdots \leq \alpha^r_1$ due to the subset inclusions, and therefore
$$\alpha^0_1 \leq \cdots \leq \alpha^r_1 \leq \beta_{n+1} \leq \cdots \leq \beta_{r+s}. \tag{5.2.41}$$
Let, then, $I \in \{-1,0,\ldots,r+s\}$ be the the smallest index such that $\alpha^{I'}_1 > j$ or else $\beta_{I'} > j$ for all $I' \geq I+1$. The adverb 'else' is warranted by (5.2.41). If the former, this implies that $N_{\alpha^{I'}} = 1$, and so $\alpha^{I'}_1 > j$, for all $I' \geq I$. Now Lemma 5.2.29 and Corollary 5.2.30 imply that (the 1-skeleton of) $\mathbf{U}^{\triangle}_{(\gamma,j)}(X)$

must be of type

$$\begin{aligned}&\mathfrak{N}_{\alpha^0_{N-1}-1} \oplus (\gamma, j)|_{\alpha^0_1-1} \to \cdots \to \mathfrak{N}_{\alpha^I_{N-1}-1} \oplus (\gamma, j)|_{\alpha^I_1-1} \to \\ &(\gamma, j)|_{\alpha^{I+1}_1-1} \to \cdots \to (\gamma, j)|_{\alpha^r_1-1} \to \\ &\gamma|_{\beta_{r+1}-1} \to \cdots \to \gamma|_{\beta_{r+s}-1}.\end{aligned}$$

If $I = -1$, then $\alpha^i_1 - 1 \geq j$ for all $i \geq 0$ and so every $(\gamma, j)|_{\alpha^i_1-1}$ is upper. We set

$$\mathbf{U}^{\triangle}_{(\gamma,j)}(X) \coloneqq \Xi^*(\gamma) \tag{5.2.42}$$

using the map

$$\begin{aligned}\Xi\colon [r+s] &\to [k+1],\\ i &\mapsto \begin{cases}\alpha^i_1 - 1, & i \leq r\\ \beta_i - 1, & r+1 \leq i \leq r+s\end{cases}\end{aligned}$$

which is monotone by (5.2.41), and observe that it is of the desired type. One such case occurs when $X = \partial$ (with $r = -1$ and $s = k+1$) and reproduces $\mathbf{U}^{\triangle}_{(\gamma,j)}(\partial) = d_{k+1}(\gamma)$.

If $I \geq 0$, then the construction is naturally partitioned into cases depending on (γ, j). Since $k+1$ is the final vertex, $d_{k+1}\gamma$ is either low or vertical.

It is low iff $j = k+1$, in which case $I = r+s$. This implies that $\mathbf{U}^{\triangle}_{(\gamma,k+1)}(X)$ has no terms of type $(\gamma, j)|_*$, and that $\gamma|_{\beta_t-1} = \mathfrak{N}_{\beta_t-1} \oplus \pi(\gamma|_{\beta_t-1})$ (as noted in the proof of Corollary 5.2.30). Similarly, we have $(\gamma, k+1)|_{\alpha^t_1-1} = \pi(\gamma|_{\alpha^t_1-1})$. In sum, $\mathbf{U}^{\triangle}_{(\gamma,k+1)}(X)$ is to be of type

$$\begin{aligned}&\mathfrak{N}_{\alpha^0_{N-1}-1} \oplus \pi(\gamma|_{\alpha^0_1-1}) \to \cdots \to \mathfrak{N}_{\alpha^r_{N-1}-1} \oplus \pi(\gamma|_{\alpha^r_1-1}) \to \\ &\mathfrak{N}_{\beta_{r+1}-1} \oplus \pi(\gamma|_{\beta_{r+1}-1}) \to \cdots \to \mathfrak{N}_{\beta_{r+s}-1} \oplus \pi(\gamma|_{\beta_{r+s}-1}).\end{aligned}$$

Let us write

$$\delta^T \coloneqq \begin{cases}\alpha^T_{N-1} - 1, & 0 \leq T \leq r\\ \beta_T - 1, & r+1 \leq T \leq r+s.\end{cases}$$

We claim that

$$\mathbf{U}^{\triangle}_{(\gamma,k+1)}(X) \coloneqq \Gamma_W \oplus \Gamma_V \tag{5.2.43}$$

does the job, where, first,

$$\begin{aligned}\Gamma_W\colon \Delta^{r+s} &\to BO(m)\\ (t_0, \ldots, t_{r+s}) &\mapsto \mathfrak{N}\left(\sum_{\delta^T=0} t_T, \ldots, \sum_{\delta^T=k} t_T\right)\end{aligned}$$

where an empty sum is understood to give 0. In other words, t_T is a summand of (and only of) entry δ^T. We observe that Γ_W is well-defined: since the exit index in this case is $j = k+1$, $\mathfrak{N}$ is a k-path. Immediately from the definition of I, we have $\delta^T = \alpha^T_{N-1} - 1 \leq j - 1 = k$ for $0 \leq T \leq r$, and similarly $\delta^T = \beta_T - 1 \leq j - 1 = k$ for $r+1 \leq T \leq r+s$, so that the coordinate expression makes sense. Finally, the sum of all entries is equal to $\sum_{T=0}^{r+s} t_T = 1$ since, by construction, every t_T appears therein exactly once.

Secondly for (5.2.43), we set

$$\Gamma_V \coloneqq \Xi^* \pi(d_{k+1}\gamma)$$

using the map $\Xi\colon [r+s] \to [k]$, $i \mapsto \begin{cases} \alpha_1^i - 1, & i \leq r \\ \beta_i - 1, & r+1 \leq i \leq r+s \end{cases}$ as with (5.2.42) except that the target is different.

The compatibility of (5.2.43) with the inductive assumption (5.2.27) holds trivially. As for (5.2.28), we may assume, without loss of generality, that $\alpha_1^T = i$ for some arbitrary but fixed $i \in \{1, \ldots, j\} = \{1, \ldots, k+1\}$ for all T, and it suffices to consider the restriction of X along $\Delta\{0, \ldots, r\} \hookrightarrow \Delta[r+s]$, since for the range from $r+1$ to $r+s$ we would have to assume $\beta_T = i$ for all T as well, whence that range is, for our purposes in this case, degenerate. We may now decompose as

$$X|_{0,\ldots,r} = \left(\underline{\alpha^0, k+2} \geq \cdots \geq \underline{\alpha^r, k+2}\right) \cup \underline{0, i}$$

with the left factor in $\mathrm{N}_\bullet\left(P_{i,k+2}^{\mathrm{op}}\right)$, so that compatibility comes into question. Let, then, $T \in [r]$ and observe that

$$\Gamma_W|_T(1) = \Gamma_W(0, \ldots, 1, \ldots, 0) = \mathfrak{N}|_{\alpha_{N-1}^T - 1}(1) = \mathrm{Op}\mathfrak{N}|_T(1),$$

by Remark 5.2.23, where in the second term 1 appears in entry T.[8] More generally, given $(t_0, \ldots, t_r) \in \Delta^r$, note that $\alpha_{N-1}^T \geq \alpha_{N-1}^{T'}$ for $T \leq T'$ as we know from the proof of Lemma 5.2.22, and so

$$\begin{aligned} \Gamma_W|_{0,\ldots,r}(t_0, \ldots, t_r) &= \Gamma_W(t_0, \ldots, t_r, 0, \ldots, 0) \\ &= \mathfrak{N}|_{\alpha_{N-1}^r - 1, \ldots, \alpha_{N-1}^0 - 1}\left(\sum_{\alpha_{N-1}^T = \alpha_{N-1}^r} t_T, \ldots, \sum_{\alpha_{N-1}^T = \alpha_{N-1}^0} t_T\right) \\ &= \mathrm{Op}\mathfrak{N}|_{0,\ldots,r}(t_0, \ldots, t_r) \end{aligned}$$

directly by the definition of the latter in Construction 5.2.21 and by Remark 5.2.23, since a poset map $\phi\colon [K] \to [L]$ is mapped under geometric realisation (see Chapter 2) to the map

$$(t_0, \ldots, t_K) \mapsto \left(\sum_{\phi(T)=0} t_T, \ldots, \sum_{\phi(T)=L} t_T\right).$$

We must also check compatibility with the condition $\mathbf{U}^{\triangle}(\partial) = d_{k+1}(\gamma)$, but this is straightforward. The relevant range is from $r+1$ to $r+s$, and here we observe that

$$\begin{aligned} \Gamma_W|_{r+1,\ldots,r+s}(t_{r+1}, \ldots, t_{r+s}) &= \Gamma_W(0, \ldots, 0, t_{r+1}, \ldots, t_{r+s}) \\ &= \mathfrak{N}|_{\beta_{r+1}-1, \ldots, \beta_{r+s}-1}(t_{r+1}, \ldots, t_{r+s}) \end{aligned}$$

[8] This 1 is the one of $\Delta^0 = \{1\} \subset \mathbf{R}$.

since β_* is non-decreasing. This implies

$$\begin{aligned}\Gamma_W \oplus \Gamma_V &= \mathfrak{N}_{\beta_{r+1}-1,\dots,\beta_{r+s}-1} \oplus \pi(d_{k+1}\gamma)|_{\beta_{r+1}-1,\dots,\beta_{r+s}-1} \\ &= d_{k+1}\gamma|_{\beta_{r+1}-1,\dots,\beta_{r+s}-1}\end{aligned}$$

since this face is low in this case by assumption.

If, lastly, $d_{k+1}(\gamma)$ is vertical, or equivalently $j \leq k$ which covers the case $0 \leq I < n+m$, then

$$d_{k+1}(\gamma, j) = (d_{k+1}\gamma, j) \tag{5.2.44}$$

since $k+1 > j$ implies $\flat_{j,k+1} = j$. We first claim that

$$\mathbf{U}_{(\gamma,j)}(\underline{0, \alpha^T, k+2}) = \begin{cases} \mathbf{U}_{d_{k+1}(\gamma,j)}(\underline{0, \alpha^T, k+1}), & \alpha^T_{r_T} \leq k, \\ \mathbf{U}_{d_{k+1}(\gamma,j)}(\underline{0, \alpha^T}), & \alpha^T_{r_T} = k+1 \end{cases} \tag{5.2.45}$$

Indeed, recall that by (5.2.16) from the proof of Lemma 5.2.15 we have

$$\mathbf{U}_{(\gamma,j)}(\underline{i, \ell}) = \mathbf{U}_{(\gamma,j)}(\underline{i, \ell'})$$

whenever $1 \leq i \leq j < \ell, \ell' \leq k+2$. This immediately implies the claim since by (5.2.44) the exit indices on both sides coincide. Namely, the latter implies

$$N_{\underline{\alpha^T, k+2}} = \begin{cases} N_{\underline{\alpha^T, k+1}}, & \alpha^T_{r_T} \leq k \\ N_{\alpha^T}, & \alpha^T_{r_T} = k+1 \end{cases}$$

since the first index on either side that exceeds $j \leq k$ is not affected by whether the full sequence ends with $k+1$ or $k+2$. Therefore, if N (which we can thus employ unambiguously) is 1, then, by Lemma 5.2.29, (5.2.45) becomes

$$(\gamma, j)|_{\alpha^T_1 - 1} = (d_{k+1}(\gamma, j))|_{\alpha^T_1 - 1} = (d_{k+1}\gamma, j)|_{\alpha^T_1 - 1} \tag{5.2.46}$$

which holds by simpliciality since $\alpha^T_1 - 1 \leq k$ as $\alpha^T \subseteq [1, k+1]$, whence $\alpha^T_1 - 1 \in \mathrm{Im}(\partial_{k+1})$. If $N > 1$, then again using Lemma 5.2.29 together with (5.2.46) we see that (5.2.45) is tantamount to

$$\mathfrak{N}(\gamma, j)(\underline{\alpha^T, k+2}_{N-1} - 1) = \begin{cases} \mathfrak{N}(d_{k+1}(\gamma, j))(\underline{\alpha^T, k+1}_{N-1} - 1), & \alpha^T_{r_T} \leq k, \\ \mathfrak{N}(d_{k+1}(\gamma, j))(\alpha^T_{N-1} - 1), & \alpha^T_{r_T} = k+1. \end{cases}$$

which holds similarly. In the same way we obtain

$$\mathbf{U}_{(\gamma,j)}(\underline{0, \beta_T, k+2}) = \begin{cases} \mathbf{U}_{d_{k+1}(\gamma,j)}(\underline{0, \beta_T, k+1}), & \beta_T \leq k \\ \mathbf{U}_{d_{k+1}(\gamma,j)}(\underline{0, k+1}), & \beta_T = k+1 \end{cases} \tag{5.2.47}$$

using Corollary 5.2.30, which is to say, as a special case of the above.

Now, since $d^{\mathcal{V}^{\hookrightarrow}}_{k+1} = d^{\mathcal{B}^{\oplus}\mathrm{O}}_{k+2}$ by Lemma 4.3.4, the inductive hypothesis provides

$$\mathbf{U}_{\leq k}(d_{k+1}(\gamma, j)) \colon \mathrm{Path}[k+1] \to B^{\oplus}\mathrm{O}$$

and so in particular a map

$$\mathbf{U}_{\leq k}(d_{k+1}(\gamma, j))|_{\mathrm{Hom}(0,k+1)} \colon \mathrm{N}_{\bullet}\left(P^{\mathrm{op}}_{0,k+1}\right) \to B\mathrm{O}(n+m). \tag{5.2.48}$$

Consider the projection

$$\Pi\colon \overline{P^{\mathrm{op}}_{0,\widehat{k+2}}} \twoheadrightarrow \overline{P^{\mathrm{op}}_{0,\widehat{k+1}}},$$

$$\underline{0,\alpha,k+2} \mapsto \begin{cases} \underline{0,\alpha,k+1}, & \alpha_n \leq k \\ \underline{0,\alpha}, & \alpha_n = k+1 \end{cases}$$

with $\alpha = (\alpha_1 \leq \cdots \leq \alpha_n)$, $n \geq 0$, within $[1, k+1]$, where $n = 0$ is understood to give the empty sequence. It is clearly functorial. Using the adjunction between geometric realisation and the singular chains functor one more time, we can write (5.2.48) as a continuous map of type $\left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+1}\right)\right| \to B\mathrm{O}(n+m)$, and, applying Lemma 5.2.37, obtain the restriction

$$\left|\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+1}}}\right)\right| \hookrightarrow \left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+1}\right)\right| \to B\mathrm{O}(n+m).$$

Finally, un-applying the adjunction yields the further-restricted ∞-functor

$$\mathbf{U}_{\leq k}(d_{k+1}(\gamma, j))\colon \mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+1}}}\right) \to B\mathrm{O}(n+m).$$

We can thus compose and obtain

$$\Pi^*\mathbf{U}_{\leq k}(d_{k+1}(\gamma, j))\colon \triangle \twoheadrightarrow \mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k+1}}}\right) \to B\mathrm{O}(n+m)$$

and consequently set

$$\mathbf{U}^{\triangle}_{(\gamma,j)}(X) \coloneqq \Pi^*\mathbf{U}_{\leq k}(d_{k+1}(\gamma, j))(X). \tag{5.2.49}$$

The equalities (5.2.45) and (5.2.47) state precisely that (5.2.49) is compatible with $\mathbf{U}_{\leq k}$.

On the other hand, we may append the compatibility of (5.2.43) and $\mathbf{U}_{\leq k}$ to the inductive hypothesis. Namely, we assume that *the map induced (by repeated use of the adjunction between geometric realisation and the singular chains functor) by* $\mathbf{U}_{\leq k}$ *itself on* $\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{k'+2}}}\right)$*, for all* $k' \leq k-1$*, is given by* (5.2.43) *on any exit path whose* $(k'+1)$*-face is low.*

This assumption is justified since it holds in the base case $k = 1$, as we will now observe. The case of interest is where the exit index is $j = 2$, and there we have given the filler of (5.2.14), which depicts exactly $\overline{P^{\mathrm{op}}_{0,\widehat{3}}}$, by $\Gamma \oplus s_0(\gamma_V)$ and we see that

$$\Gamma(t_0, t_1, t_2) = \gamma_W(t_1, 1-t_1) = \gamma_W(t_1, t_0 + t_2) = \Gamma_W(t_0, t_1, t_2).$$

Similarly, on $X = (\underline{0123} \geq \underline{013} \to \underline{023})$ we have $\Xi\colon [r+s] = [0+2] \to [1]$ maps $0 \mapsto \alpha^0_1 - 1 = 0$, $1 \mapsto \beta_1 - 1 = 0$, $2 \mapsto \beta_2 - 1 = 1$, and so $\Xi = \sigma_0$. Thus

$$s_0(\gamma_V) = \Xi^*\pi(d_2\gamma) = \Gamma_V.$$

Consequently, (5.2.49) itself is also compatible with (5.2.43) by virtue of being compatible with $\mathbf{U}_{\leq k}$ by the inductive assumption. $\square$

The proof of Proposition 5.2.39 cannot be read off from the examples in Section 5.2.2 and Example 5.2.33 alone. Let us therefore give two final examples that illustrate the novel cases treated in that proof.

Example 5.2.50. Let us consider a case where $d_{k+1}\gamma$ is vertical. Suppose $k = 2$ and $j = 2$, so that γ (as visualised within the 3-cylinder) is of type

$$\begin{tikzcd}
 & K \arrow[dr] & \\
 & & K' \\
 & W_1 \oplus V_1 \arrow[uu] \arrow[ur] & \\
W_0 \oplus V_0 \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

where we omitted the edge $W_0 \oplus V_0 \to K'$. The face $d_{k+1}\gamma$ is

$$\begin{tikzcd}
 & W_1 \oplus V_1 \arrow[dr] & \\
W_0 \oplus V_0 \arrow[ur] \arrow[rr] & & K
\end{tikzcd}.$$

The image of $P^{\mathrm{op}}_{0,4}$ under $\mathbf{U}$, with γ painted in blue into ∇ within the geometric realisation, is then of type

$$\begin{tikzcd}
 & W_1 \oplus V_0 \arrow[rr] \arrow[dl] \arrow[dd, equal] & & W_1 \oplus V_0 \arrow[dl] \arrow[dd] \\
W_1 \oplus V_1 \arrow[rr] \arrow[dd] & & W_1 \oplus V_1 \arrow[dd, blue] \arrow[ddll, blue] & \\
 & W_0 \oplus V_0 \arrow[rr] \arrow[dl] & & W_0 \oplus V_0, \arrow[ul, blue] \arrow[dl, blue] \arrow[dlll, blue] \\
K \arrow[rr, blue] & & K' &
\end{tikzcd}.$$

A path X in $\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\hat{4}}}\right)$ as in (5.2.40) that is novel is given, for instance, by $\underline{0124} \geq \underline{024} \to \underline{034}$, whose image under $\mathbf{U}^{\triangle}$, as can be read off the cube, is to be of type

$$W_1 \oplus V_0 \to W_1 \oplus V_1 \to K.$$

The filler thereof provided by the proof is exhibited by first factoring X through the projection Π, which yields $\Pi(X) = (\underline{0123} \to \underline{023} \to \underline{03})$, which by $\mathbf{U}(d_3(\gamma, 2))$ is mapped to a 2-simplex of type

$$\begin{tikzcd}
 & W_1 \oplus V_1 \arrow[dr] & \\
W_1 \oplus V_0 \arrow[ur] \arrow[rr] & & K
\end{tikzcd}$$

provided exactly by the left triangle in (5.2.11) in the case $j = 2 = \flat_{2,3}$.

Example 5.2.51. In the situation of Example 5.2.33, i.e., with $k = 2$ and $j = 3$ so that $d_{k+1}\gamma$ is low, a novel path X in $\mathrm{N}_\bullet\left(\overline{P^{\mathrm{op}}_{0,\widehat{4}}}\right)$ as in (5.2.40) is given, for instance, by $\underline{01234} \geq \underline{0124} \geq \underline{024} \to \underline{034}$. Its image under $\mathbf{U}^\triangle$ is to be of type

$$W_2 \oplus V_0 \to W_1 \oplus V_0 \to W_1 \oplus V_1 \to W_2 \oplus V_2. \tag{5.2.52}$$

We read off

$$\Gamma_W(t_0, t_1, t_2, t_3) = \gamma_W(0, t_1 + t_2, t_0 + t_3)$$

where the normal path γ_W is of type

$$\begin{tikzcd} & W_1 \arrow[dr] & \\ W_0 \arrow[ur] \arrow[rr] & & W_2 \end{tikzcd}.$$

It is a direct check to see that Γ_W fills the normal component of (5.2.52).

We will conclude the present chapter with a construction promised in Remark 4.3.14.

Remark 5.2.53. Let $\Gamma\colon \mathrm{Path}[k+1] \to B^\oplus\mathrm{O}$ be a k-simplex of $\mathcal{V}^\hookrightarrow$, and $\triangledown$ as in Construction 5.2.31. Then the rule[9]

$$\begin{aligned} \mathcal{V}^\hookrightarrow_k &\to (B\mathrm{O}^\infty_{\amalg})_k \\ \Gamma &\mapsto \Gamma(\triangledown), \end{aligned}$$

restricts on the core $\mathcal{V}^\simeq \subset \mathcal{V}^\hookrightarrow$ to an inverse to the map $\Psi\colon B\mathrm{O}^\infty_\amalg \to \mathcal{V}^\hookrightarrow$ from the proof of Theorem 4.3.11. In contrast to the putative rule Ψ^{-1} in Remark 4.3.14, this is functorial: $\triangledown = \triangledown^k\colon \Delta^k \hookrightarrow \left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+1}\right)\right|$ is the convex hull of the corners $\underline{0, i, k+1}$ and $\underline{0, k+1}$ within the k-cube (along Proposition 5.2.18), so evidently $(d^{\mathcal{B}^\oplus\mathrm{O}}_{i+1}\Gamma)(\triangledown^{k-1}) = d_i\Gamma(\triangledown^k)$ and $(s^{\mathcal{B}^\oplus\mathrm{O}}_{i+1}\Gamma)(\triangledown^{k+1}) = s_{i+1}\Gamma(\triangledown^k)$ hold for $i \in \{0, \ldots, k\}$, which by Lemma 4.3.4 gives simpliciality.

Remark 5.2.54. None of the results and constructions in this section depends on the properties of infinite Grassmannians. Conseqently, for any topological monoid M as in Corollary 4.3.15, assume that its operation $\odot$ is a cofibration, and that $M = \coprod M_i$ over some index set. Then we may consider two 'strata' M_1, M_2 and the restricted operation $\odot\colon M_1 \times M_2 \to M_{i_{12}}$, giving the linked space $(M_1 \leftarrow M_1 \times M_2 \to M_{i_{12}})$. Then the unpacking map gives a fully-faithful[10] ∞-functor

$$\mathcal{E}\mathcal{X}\left(M_1 \leftarrow M_1 \times M_2 \to M_{i_{12}}\right) \to */\mathrm{N}^{\mathrm{hc}}(BM),$$

[9]where we use the restriction $\Gamma\colon \mathrm{Hom}_{\mathrm{Path}[k+1]}(0, k+1) = \mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+1}\right) \to B\mathrm{O}^\infty_\amalg$, take the corresponding continous map $\left|\mathrm{N}_\bullet\left(P^{\mathrm{op}}_{0,k+1}\right)\right| \to B\mathrm{O}^\infty_\amalg$, and finally pull it back along $\triangledown\colon \Delta^k \hookrightarrow |\mathrm{N}_\bullet|$

[10]as noted in the proof of Proposition 6.3.13.

giving a particularly simple equivalent description of the full sub-∞-categories of the 'quasi-deloop-and-loop' of $(M, \odot)$ generated by depth-1 pairs.

CHAPTER 6

Cartesian structures

Definition. A linked space $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ is called a *linked manifold* if

- M, L, and N are smooth manifolds,
- π is a fibre bundle,[1] and
- ι is a closed embedding.

6.1. Linked tangent bundles

We start by observing a fact, Lemma 6.1.1, that will let us give the tangent bundle of a linked manifold by means of unstratified data. This formalises an informal discussion present in [7, §2.1.4] in the conically-smooth setting.

From now on, we assume all manifolds Hausdorff and paracompact, so that vector sub-bundles split.

Let $\iota\colon L \hookrightarrow N$ be a clossed embedding of smooth manifolds, and $E \to N$ a rank-$(n+m)$ vector bundle classified by $E\colon N \to BO(n+m)$, equipped with the inner product induced by that on the separable Hilbert space $H \cong \mathbf{R}^\infty$ used to construct the Grassmannians $BO(k) = \mathrm{Gr}_k(H)$. Let further E_0 be a rank-n vector sub-bundle of ι^*E, classified by $E_0\colon L \to BO(n)$. The pullback bundle itself is classified by $\iota^*E\colon L \hookrightarrow N \to BO(n+m)$.

The normal bundle $E_0^\perp \subset \iota^*E$, classified by $E_0^\perp\colon L \to BO(m)$, satisfies $E_0 \oplus E_0^\perp \cong \iota^*E$. It is classical that the Whitney sum is classified as follows: Consider the isomorphism

$$\Phi\colon H \oplus H \cong H$$

given by sending, with respect to a fixed basis of H indexed over $\mathbf{N}$, the first copy to odd coordinates and the second copy to even coordinates. The (abstract) direct sum precomposes with this isomorphism to give a map

$$\begin{aligned} E_0^\perp \widetilde{\oplus} E_0\colon L &\xrightarrow{E_0 \times E_0^\perp} \mathrm{Gr}_n(H) \times \mathrm{Gr}_m(H) \\ &\xrightarrow{\oplus} \mathrm{Gr}_{n+m}(H \oplus H) \\ &\overset{\Phi}{\cong} \mathrm{Gr}_{n+m}(H) = BO(n+m). \end{aligned}$$

The classifier

$$\oplus_W\colon L \to BO(n+m)$$

of the Whitney sum $E_0 \oplus E_0^\perp$ is then homotopic to $E_0^\perp \widetilde{\oplus} E_0$.

[1] For the construction of the tangent bundle of a linked manifold (Construction 6.1.5), it is enough that π be a surjective submersion.

Lemma 6.1.1. *Let*

- $\iota\colon L \hookrightarrow N$ *be a closed embedding of smooth manifolds,*
- $E \to N$ *a rank-*$(n+m)$ *vector bundle equipped with an inner product,*
- *and* $E_0 \hookrightarrow \iota^* E$ *a rank-*n *vector sub-bundle.*

Then there exists a classifier $E\colon N \to \mathrm{Gr}_{n+m}(H \oplus H)$ *of the isomorphism class of* $E \to N$ *such that the diagram*

$$\begin{tikzcd} L \arrow[r, "E_0 \times E_0^\perp"] \arrow[d, hook, "\iota"'] & \mathrm{Gr}_n(H) \times \mathrm{Gr}_m(H) \arrow[d, "\oplus"] \\ N \arrow[r, "E"'] & \mathrm{Gr}_{n+m}(H \oplus H) \end{tikzcd} \tag{6.1.2}$$

commutes.[2]

Proof. Let us concatenate the homotopy $\oplus_W \sim E_0^\perp \widetilde{\oplus} E_0$ constructed above with the standard one from $\iota^* E$ to $\oplus_W$, classifying the inverse of the bundle isomorphism $E_0 \oplus E_0^\perp \cong \iota^* E$ given fibrewise by $(v, w) \mapsto v + w$, to obtain a homotopy

$$h\colon \iota^* E \to \oplus_W \to E_0^\perp \widetilde{\oplus} E_0,$$

of maps $L \to BO(n+m)$, which sits in the commutative diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, hook, "\iota"'] & BO(n+m)^I \arrow[d, "\mathrm{ev}_0"] \\ N \arrow[r, "E"'] \arrow[ur, dotted, "H"] & BO(n+m) \end{tikzcd}.$$

As ι, being a closed embedding, is a cofibration, there exists a homotopy extension $H\colon N \to BO(n+m)^I$ as depicted. We may now consider

$$E' \coloneqq H_1\colon N \to BO(n+m)$$

and apply the inverse isomorphism $\Phi^{-1}\colon H \cong H \oplus H$ to obtain

$$\Phi^{-1} E'\colon N \to \mathrm{Gr}_n(H \oplus H).$$

On the other hand, applying Φ^{-1} to $E_0^\perp \widetilde{\oplus} E_0$ recovers $E_0^\perp \oplus E_0 = \oplus \circ E_0 \times E_0^\perp$. Therefore the two classifiers

$$E_0^\perp \oplus E_0, \iota^* E'\colon L \to \mathrm{Gr}_{n+m}(H \oplus H)$$

coincide. $\square$

Notation 6.1.3. We will sometimes write simply $BO(k)$ for $\mathrm{Gr}_k(H^{\oplus -}) \cong BO(k)$ for any countable number of copies of H, and therefore abuse notation in diagrams of type (6.1.2).

Let now $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked manifold, with each manifold riemannian. As above, they are all assumed paracompact, while Hausdorffness is automatic. Given this contractible choice of metrics, we will show that there

[2]The point being that it doesn't just homotopy-commute.

is a 'canonical' map

$$\mathrm{T}\mathfrak{S}\colon \mathfrak{S} \to B\mathrm{O}(n, m) \tag{6.1.4}$$

of linked spaces, which we call the *tangent bundle* of $\mathfrak{S}$.

Construction 6.1.5. Since $\mathrm{d}\pi$ surjects, the induced linear dual map

$$(\pi^*\mathrm{T}M)^\vee \hookrightarrow (\mathrm{T}L)^\vee,$$

of bundles over L, injects. Using the metrics, this gives an injection

$$\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L.$$

Composing with $\mathrm{d}\iota$, we have a bundle injection

$$\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L \hookrightarrow \iota^*\mathrm{T}N$$

over L. Let us denote the normal bundle of this injection by

$$\mathrm{N} \coloneqq \mathrm{N}_N M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \iota^*\mathrm{T}N.$$

Now, in the diagram

$$\begin{tikzcd}
L \arrow[rr, "\pi^*\mathrm{T}M\times\mathrm{N}"] \arrow[dr, hook, "\iota"] \arrow[dd, two heads, "\pi"'] & & B\mathrm{O}(n)\times B\mathrm{O}(m) \arrow[dr, "\oplus"] \arrow[ddl, "\mathrm{pr}"'] & \\
& N \arrow[rr, "\mathrm{T}N"] & & B\mathrm{O}(n+m) \\
M \arrow[r, "\mathrm{T}M"] & B\mathrm{O}(n) & &
\end{tikzcd} , \tag{6.1.6}$$

the back square

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & B\mathrm{O}(n)\times B\mathrm{O}(m) \arrow[d] \\
N \arrow[r] & B\mathrm{O}(n+m)
\end{tikzcd}$$

commutes using Lemma 6.1.1 and Notation 6.1.3, and the front square commutes trivially. This yields the span map (6.1.4).

Remark 6.1.7. Writing

$$\mathrm{N}_L M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \mathrm{T}L,$$

we have a splitting

$$\mathrm{T}L \cong \pi^*\mathrm{T}M \oplus \mathrm{N}_L M.$$

Similarly, writing

$$\mathrm{N}_N L \coloneqq (\mathrm{T}L)^\perp \subset \iota^*\mathrm{T}N,$$

we have a splitting

$$\iota^*\mathrm{T}N \cong \mathrm{T}L \oplus \mathrm{N}_N L \cong \pi^*\mathrm{T}M \oplus \mathrm{N}_L M \oplus \mathrm{N}_N L.$$

Thus

$$\mathrm{N}_N M \cong \mathrm{N}_L M \oplus \mathrm{N}_N L.$$

In practice, the bundle N is best determined in two steps via this decomposition.

Applying $\mathcal{EX}$ and post-composing with $\mathbf{U}$, we have the induced map

$$\mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow},$$

which is the linked version of (the classifying map of) the (conically-smooth) constructible tangent bundle.

Example 6.1.8. If L is induced by a closed submanifold inclusion $M \subset \overline{N}$ as $L = \mathbb{S}(\mathrm{N}_N M)$, the sphere bundle of the normal bundle (Example 3.2.16), then L has dimension $n+m-1$, $\mathrm{N}_L M$ has rank $m-1$, and $\mathrm{N}_L N$ has rank 1. More specifically, in the conically-smooth context, the link (of a pair of strata) comes with an open embedding $L \times \mathbf{R} \hookrightarrow N$, which is tantamount to the triviality of the latter normal bundle, i.e., $\mathrm{N}_N L \simeq \varepsilon^1$, or, equivalently, to a diffeomorphism $L \times \mathbf{R} \simeq \mathbb{S}\left(\mathrm{N}_N M\right) \times \mathbf{R}$. This $\mathbf{R}$-factor incarnates the extra $\mathbf{E}_1$-structure featuring in the classification of stratified locally-constant (a.k.a. *constructible*) factorisation algebras on stratified spaces of type $M \subset \overline{N}$.

Example 6.1.9. An even simpler situation arises when L (and $\mathfrak{S}$) is induced by a boundary $M = \partial\overline{N} \subset \overline{N}$ as $L \cong M$, the boundary pushed diffeomorphically into the interior $N = \overline{N} \setminus M$ by following the flow of a nowhere-vanishing inward pointing vector field along the boundary (which always exists) for a chosen non-zero time (Example 3.2.15); we will denote this closed link embedding later by ι_+. Then $\mathrm{N}_L M = 0$ and $\mathrm{N}_N L \simeq \varepsilon^1$ again.

Definition 6.1.10. We call a linked manifold with M of dimension n and N of dimension $n+m$ *constructible* if L is of dimension $n+m-1$, and its normal bundle in N is trivial.

Making the linked tangent bundle (6.1.6) an on-the-nose span map may be justified by the fact that the only real homotopy involved in (the proof of) Lemma 6.1.1 is the classical one between $\iota^* E$ and $E_0^{\perp} \widetilde{\oplus} E_0$ over L, which is canonical in the sense that it does not depend on E or E_0. This choice contains no geometric information, so it would be unwise to change the (n,m)-Grassmannian by taking a replacement only to remain agnostic about it. Besides, from a more practical point of view, the map $\mathbf{U}\colon \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}$ is natural only for this span $B\mathrm{O}(n,m)$.

6.2. Adapting AFR-type structures

The ∞-category of *tangential structures* is the over-∞-category

$$\mathcal{C}\mathrm{at}_\infty / \mathcal{V}^{\hookrightarrow}$$

as per [7].[3] Via

$$\mathbf{U}^*\colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(B\mathrm{O}(n,m)),$$

these transfer to tangential structures on linked manifolds: given $\mathfrak{S}$, and writing $\mathcal{B}_{(n,m)} \coloneqq \mathbf{U}^*\mathcal{B}$, we may define the space (homotopy type) of $\mathcal{B}$-structures on $\mathfrak{S}$ to be

$$\mathcal{B}\text{-red}(\mathfrak{S}) \coloneqq \mathrm{Map}_{/B\mathrm{O}(n,m)}\left(\mathcal{EX}(\mathfrak{S}), \mathcal{B}_{(n,m)}\right), \tag{6.2.1}$$

[3]See [51, §3] for the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

the mapping space in $\mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(B\mathrm{O}(n,m))$, where the first argument uses $\mathrm{T}\mathfrak{S}$ (Construction 6.1.5). Equivalently,

$$\mathcal{B}\text{-red}(\mathfrak{S}) = \Gamma\left((\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)}\right),$$

the homotopy-sections of $(\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)} \to \mathcal{EX}(\mathfrak{S})$.

Given $\mathcal{B}$ and (n,m), a natural question is whether

$$\mathcal{B}_{(n,m)} = \mathcal{EX}(\mathfrak{B}) \tag{6.2.2}$$

for a linked space $\mathfrak{B}$, which would enable us to discuss stratified tangential structures without having to refer to exit paths. We will restrict ourselves in this work to the case where $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ is induced by a smooth tangential structure by a cartesian fibration replacement, defined in Section 6.3.

The reason we consider this problem at all is that such tangential structures are central to our considerations in Chapter 7, where we consider linked spaces induced by bordisms with defects, equipped mostly with stable replacements of smooth tangential structures, which are refinements of cartesian structures. Besides applications, this is the main theoretical reason for our restriction to cartesian structures: for arbitrary stratified tangential structures, it is not clear how to define normal bundles, if this is at all possible: see Remark 7.7.1.

With this restriction, we give in Section 6.3 a solution to problem (6.2.2) for smooth $\mathfrak{S}$, i.e., consisting of a single stratum. Then, in Section 6.4, we will in fact see that the problem as stated is a bit too restrictive for arbitrary $\mathfrak{S}$. We identify instead simply a span $\mathfrak{B}$ of spaces that does the job just as well: see Observation 6.4.19.

We will first discuss the simplest example. To begin with, recall that for $\kappa \in \mathbf{N}$, rank-κ framings (*κ-framings*) are expressed by the tangential structure $\kappa\colon * \to \mathcal{V}^{\hookrightarrow}$ that sends the point to $\kappa \coloneqq \mathbf{R}^\kappa$.

Example 6.2.3 (framings)**.** We have

$$\kappa_{(n,m)} = \begin{cases} \mathcal{EX}\left(\emptyset \leftarrow \emptyset \to *\right) = *, & n+m=\kappa, \\ \emptyset, & \text{else} \end{cases}$$

with

$$\left(\kappa_{(n,m)} \to \mathcal{EX}\left(B\mathrm{O}(n,m)\right)\right) = \mathcal{EX}\left((\emptyset \leftarrow \emptyset \to *) \to B\mathrm{O}(n,m), * \xrightarrow{\kappa} B\mathrm{O}(n+m)\right).$$

This reflects the fact that a nontrivially stratified space does not admit a κ-framing: the else-statement implies that for a linked space to admit a κ-framing its bulk must be κ-dimensional. The first statement implies moreover that for a lift of $\mathrm{T}\mathfrak{S}$ to $(\emptyset \leftarrow \emptyset \to *)$ to exist, the space must be of type $\mathfrak{S} = (\emptyset \leftarrow \emptyset \to N)$ (if non-empty), and $\dim N = \kappa$.

Similar considerations apply to any *smooth tangential structure* $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, i.e., one that factors through $B\mathrm{O}(\kappa) \hookrightarrow \mathcal{V}^{\simeq} \hookrightarrow \mathcal{V}^{\hookrightarrow}$ for some k.

Example 6.2.4. Let $\mathfrak{b}$ be a smooth tangential structure given by a map $B \to B\mathrm{O}(\kappa)$ of spaces, e.g., induced by a map $G \to \mathrm{O}(\kappa)$ of topological groups, or a rank-κ bundle $X \to B\mathrm{O}(\kappa)$ on a space X. Then,

$$B_{(n,m)} = \begin{cases} \mathcal{EX}\,(\emptyset \leftarrow \emptyset \to B) = B, & n+m=\kappa, \\ \emptyset, & \text{else,} \end{cases}$$

where we abbreviated $\mathrm{Sing}_\bullet(B)$ to B in its last occurence.

Example 6.2.5. Consider $\mathbf{N} = (\mathbf{N}, \leq)$ with the standard order. Variframings ([7]) are given by $\mathrm{vfr}\colon \mathbf{N} \to \mathcal{V}^{\hookrightarrow}$, $k \mapsto \mathbf{k}$, $(k \leq K) \mapsto (\mathbf{k} \xhookrightarrow{-\oplus 0} \mathbf{K})$. We read $\mathrm{vfr}(k \leq K)$ as the standard[4] isomorphism $\mathbf{k} \oplus (\mathbf{K} - \mathbf{k}) \cong \mathbf{K}$. Let us restrict vfr to depth 1 by choosing a pair $n \leq N$, i.e., consider $\mathrm{vfr}|_{n \leq N}\colon \{n \leq N\} \to \mathcal{V}^{\hookrightarrow}$. Then, for $m = N - n$, we have (cf. Corollary 3.3.3)

$$\mathbf{U}^*(\mathrm{vfr}|_{n \leq N}) \simeq \mathcal{EX}(* \leftarrow * \to *) \simeq \Delta[1],$$

the exit path ∞-category of the nontrivially-linked point. Moreover,

$$\mathbf{U}^*(\mathrm{vfr}|_{n \leq N}) \to \mathcal{EX}(B\mathrm{O}(n,m))$$

is $\mathcal{EX}$ of

$$\begin{tikzcd}
& * \arrow[dl] \arrow[d, "\mathbf{n}\times\mathbf{m}"] \arrow[dr] & \\
* \arrow[d, "\mathbf{n}"] & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\oplus"] & * \arrow[d, "\mathbf{N}"] \\
B\mathrm{O}(n) & & B\mathrm{O}(N)
\end{tikzcd}.$$

Thus, a *variframing* on $\mathfrak{S} = (M \leftarrow L \hookrightarrow N)$, i.e., a lift of $\mathrm{T}\mathfrak{S}$ to this $\Delta[1]$, is a framing on M, a framing on N, and a framing on $\mathrm{N}_N M$. As such, it is more relaxed than a stable N-framing.

Example 6.2.6 (point defects)**.** The choice of a point p in a smooth manifold N of dimension n and a coordinate neighbourhood around it induce a linked space

$$\mathfrak{N}_p \coloneqq \Big(\{p\} \leftarrow S^{n-1} \xhookrightarrow{\iota_p = \iota} N \setminus \{p\}\Big)$$

where the sphere is the unit sphere in coordinates. The link map of $\mathrm{T}\mathfrak{N}_p$ reads

$$\varepsilon^0 \times \big(\mathrm{T}S^{n-1} \oplus \mathrm{N}(\iota)\big) : S^{n-1} \to * \times B\mathrm{O}(n),$$

i.e.,

$$\iota^*\mathrm{T}\,(N \setminus \{p\}) : S^{n-1} \to B\mathrm{O}(n).$$

A $\mathrm{vfr}_{0 \leq n}$-structure on $\mathfrak{N}_p$ is a framing on N together with a framing on the normal bundle of ι. In this example, the latter always exists,[5] which is why we will call such a configuration a trivial point defect.

Two relaxations of the tangential structure κ (or of any smooth structure) are of particular interest to us. They are termed, in increasing order of generality, the *stable* and *solid* replacements.

[4] Up to, of course, the choice of a pairing function $\mathbf{N} \times \mathbf{N} \cong \mathbf{N}$

[5] The normal bundles of the unit spheres are trivial.

6.3. Cartesian replacements, I: The smooth case

For $J \in \mathbf{N}$, a *stably-J-framed* smooth manifold M of dimension n is one with a framing on $\mathrm{T}M \oplus \varepsilon^j$, $J = n + j$. In other words, this amounts to an injection

$$\mathrm{T}M \hookrightarrow \varepsilon^J$$

of bundles over M whose normal bundle, defined either using a metric on ε^J or as the quotient $\varepsilon^J/\mathrm{T}M$, is parallelised.

More generally, a *solid J-framing* on M is merely an injection $\mathrm{T}M \hookrightarrow \varepsilon^J$. First of all, we notice that in order to impose the parallelisability of the normal bundle *in terms of reductions or extensions of structure groups*, we must first separate it from the solid datum.

Let X be a smooth manifold equipped with a vector bundle $E \to X$ of rank r, and let $F \to X$ be another bundle, of rank R. Choosing a bundle embedding

$$E \hookrightarrow F$$

is a reduction or extension of gauge group on E in the following way. There is naturally a normal bundle N to E such that the embedding amounts to an isomorphism

$$\Phi\colon E \oplus N \cong F.$$

This Φ provides a filler for the diagram

$$\begin{tikzcd} & B\mathrm{O}(r) \times B\mathrm{O}(R-r) \arrow[d, "\oplus"] \\ X \arrow[ur, dotted, "E\times N"] \arrow[r, "F"'] & B\mathrm{O}(R). \end{tikzcd}$$

Changing our point of view slightly, consider the limit space[6]

$$\begin{tikzcd} X \times_{B\mathrm{O}(R)} (B\mathrm{O}(r) \times B\mathrm{O}(R-r)) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & B\mathrm{O}(r) \times B\mathrm{O}(R-r) \arrow[d, "\oplus"] \\ X \arrow[r, "F"] & B\mathrm{O}(R) \end{tikzcd} \tag{6.3.1}$$

which also admits a 'source evaluation' by projecting to the first factor:

$$\mathrm{ev}_0\colon (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \to B\mathrm{O}(r).$$

Now, writing

$$(B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \coloneqq X \times_{B\mathrm{O}(R)} (B\mathrm{O}(r) \times B\mathrm{O}(R-r)), \tag{6.3.2}$$

the choice of Φ can be expressed as follows:

[6]For the moment, we disregard the appropriate homotopy versions of such limits in order to ease notation; in terms of the tangential structure, this amounts to disregarding the choice of bundle isomorphism. Homotopy limits will be reincorporated into this account of their own accord below.

Definition 6.3.3. A *solid F-structure* or *-reduction* on E (or on X when $E = \mathrm{T}X$) of is a lift of the form

$$\begin{tikzcd} & (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[ur, dotted] \arrow[r, "E"] & B\mathrm{O}(r) \end{tikzcd}.$$

We call R the *total rank* of the solid structure.

The normal bundle itself can be recovered from such a lift by projecting it to the second factor:

$$N\colon X \to (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \to B\mathrm{O}(R-r).$$

Thus, a further, simultaneous reduction on N can be implemented using this projection: if N is to have $(B \to B\mathrm{O}(R-r))$-structure, then we may consider the iterated fibre product

$$\begin{tikzcd} (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \times_{B\mathrm{O}(R-r)} B \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner", very near start] & (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \arrow[d] \\ B \arrow[r] & B\mathrm{O}(R-r) \end{tikzcd} \tag{6.3.4}$$

and, writing

$$(B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_{(F,B)} := (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \times_{B\mathrm{O}(R-r)} B, \tag{6.3.5}$$

ask for reductions of the following form:

Definition 6.3.6. A *solid (F, B)-structure* on E (or on X when $E = \mathrm{T}X$) is a lift of the form

$$\begin{tikzcd} & (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_{(F,B)} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[ur, dotted] \arrow[r, "E"] & B\mathrm{O}(r) \end{tikzcd}.$$

When $B = \mathbf{R} - \mathbf{r}$, this is a *stable F-structure.* We call $B \to B\mathrm{O}(R-r)$, or B, the *normal structure.* When F is clear or unimportant, we also simply say *solid/stable Y-structure.*

It is incidental that F is given as a bundle over X. More generally, when

$$F\colon Y \to B\mathrm{O}(R)$$

is any smooth tangential structure with rank $R \geq r = \mathrm{rk}(E)$, the limit (6.3.1), and so also (6.3.4), still make sense. Then, a *solid Y- or (Y, B)-structure* is defined analogously, as is a *stable Y-structure.*

Warning 6.3.7. The definition of a stable structure given in Definition 6.3.6 is similar to but also completely different from another very common definition in the literature, according to which, for instance, M is stably-framed if $M \times \mathbf{R}^d$

for *some* $d \geq 0$ is framed. For us, the total rank is fixed, so there is a single candidate for the trivial factor. We will never use this variant in this work.

Solid replacements in the stratified context have been considered in [7]. In categorical terms, they are cartesian fibration replacements. Namely, the assignment in the following Definition 6.3.8 extends (by a main result of [35]) to a left-adjoint

$$\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow}$$

to the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ from *cartesian* tangential structures (i.e., cartesian fibrations over $\mathcal{V}^{\hookrightarrow}$) to tangential structures.

Definition 6.3.8. Given a tangential structure $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, its *cartesian (fibration) replacement* is

$$\overline{\mathfrak{b}}\colon \overline{\mathcal{B}} = \overline{(\mathcal{B}, \mathfrak{b})} = (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} \mathcal{B} \to (\mathcal{V}^{\hookrightarrow})^{\{0\}},$$

the source evaluation from the fibre product along the target evaluation.

Note the direct correspondence with Definition 6.3.3 (and (6.3.1)), in view of $B\mathrm{O}(r) \times B\mathrm{O}(R-r)$'s being the link of the $(r, R-r)$-Grassmannian, viewing the target evaluation as the embedding $\oplus$ off of the link. We will make this precise.

A solid $(F : Y \to B\mathrm{O}(R))$-structure on a rank-$r$ bundle ought to be (a lift to) the restriction to $B\mathrm{O}(r)$ of the solid replacement of F:

$$\begin{tikzcd} \overline{Y}|_r \coloneqq B\mathrm{O}(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} \overline{Y} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & \overline{Y} \arrow[d, "\overline{F}"] \\ B\mathrm{O}(r) \arrow[r] & \mathcal{V}^{\hookrightarrow} \end{tikzcd} . \tag{6.3.9}$$

This is the space of morphisms in $\mathcal{V}^{\hookrightarrow}$ that start in $B\mathrm{O}(r)$ and end in the image of F inside $B\mathrm{O}(R)$.

Lemma 6.3.10. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) \simeq \mathrm{Hom}_{\mathcal{EX}(B\mathrm{O}(r,R-r))}(p, q)$, *where* $p \in B\mathrm{O}(r)$ *and* $q \in B\mathrm{O}(R)$.

Proof. For the most part, we repeat the argument in Remark 4.3.2 – see there for the references. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q)$ is equivalent to the homotopy-fibre of

$$p^*\colon \mathrm{Hom}_{\mathrm{N^{hc}}(\mathcal{B}\oplus\mathrm{O})}(*, *) \to \mathrm{Hom}_{\mathrm{N^{hc}}(\mathcal{B}\oplus\mathrm{O})}(*, *),$$

at q. Morphism spaces in a homotopy-coherent nerve are equivalent to those in the original topological category, so it is equivalent to the homotopy-fibre of

$$(- \oplus p)\colon B\mathrm{O}^\infty_{\mathrm{II}} \to B\mathrm{O}^\infty_{\mathrm{II}}$$

at q. The connected component of q in $B\mathrm{O}^{\infty}_{\amalg}$ is $B\mathrm{O}(R)$, and p^* maps only $B\mathrm{O}(R-r)$ into it, so we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p,q) &= (B\mathrm{O}(R-r)\oplus p)\times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]}\times_{B\mathrm{O}(R)^{\{1\}}}\{q\}\\ &= \pi^{-1}(p)\times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]}\times_{B\mathrm{O}(R)^{\{1\}}}\{q\}\\ &= P(B\mathrm{O}(R))_{\pi^{-1}(p),q}\\ &\simeq \mathrm{Hom}_{\mathcal{EX}(B\mathrm{O}(r,R-r))}(p,q)\end{aligned}$$

by Theorem 3.3.1. By $\pi = \mathrm{pr}_1$ we denoted the link projection in $B\mathrm{O}(r,R-r)$. □

Remark 6.3.11. We should note that $\overline{Y}|_r$ is *not*, in general, the ∞-categorical homotopy fibre product

$$B\mathrm{O}(r)\times^{\mathrm{h}}_{\mathcal{V}^{\hookrightarrow}} Y = B\mathrm{O}(r)\times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}}\mathrm{Isom}(\mathcal{V}^{\hookrightarrow})\times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y$$

in the sense of [52, §01DE]: isomorphisms in $\mathcal{V}^{\hookrightarrow}$ from $B\mathrm{O}(r)$ to $F(Y)$ exist iff $r=R$. A *Kan* fibration replacement rather than a cartesian one would employ $B\mathrm{O}(r)\times^{\mathrm{h}}_{\mathcal{V}^{\hookrightarrow}} Y$.

The following is a consequence of Lemma 6.3.10 and the fully-faithfulness of $\mathbf{U}$: see the proof of Proposition 6.3.13.

Corollary 6.3.12. *$\mathcal{EX}(B\mathrm{O}(n,m))$ is equivalent to the full sub-∞-category $\mathcal{V}^{\hookrightarrow}|_{n,m}$ of $\mathcal{V}^{\hookrightarrow}$ generated by $B\mathrm{O}(n), B\mathrm{O}(n+m)\subset\mathcal{V}^{\simeq}$.*

Thus, the result of Joyal–Lurie/Hebestreit–Krause combined with Theorem 3.3.1 hints at an alternative means of providing $\mathbf{U}\colon \mathcal{EX}B\mathrm{O}(n,m)\hookrightarrow\mathcal{V}^{\hookrightarrow}$ in its topological incarnation. However, this is hard to make explicit, (not too) unlike our construction of $\mathbf{U}$ as a map of simplicial sets.

Proposition 6.3.13. $\overline{Y}|_r \simeq (B\mathrm{O}(r)\times B\mathrm{O}(R-r))\times^{\mathrm{h}}_{B\mathrm{O}(R)} Y$.

Proof. Written in full, the statement reads

$$\begin{gathered}B\mathrm{O}(r)\times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}}(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y\\ \simeq\\ (B\mathrm{O}(r)\times B\mathrm{O}(R-r))\times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]}\times_{B\mathrm{O}(R)^{\{1\}}} Y.\end{gathered}$$

By direct on inspection of $\mathbf{U}$ on exit paths of maximal index, which in the proof of Proposition 5.2.39 is the case where $d_{k+1}\gamma$ is low (the inverse is given by sending a $(k+1)$-path in $B^{\oplus}\mathrm{O}$ to the image k-simplex of ∇ under it.[7]), and by Lemma 6.3.10 we have that $\mathbf{U}\colon\mathcal{EX}=\mathcal{EX}(B\mathrm{O}(r,R-r))\to\mathcal{V}^{\hookrightarrow}$ is fully-faithful, whence it is an equivalence onto its image $\mathcal{V}^{\hookrightarrow}(r,R)$, the full sub-$\infty$-category generated by $B\mathrm{O}(r)\amalg B\mathrm{O}(R)\subset\mathcal{V}^{\hookrightarrow}$. Thus, we observe that

[7] Recall the construction from Lemma 5.2.37 and Proposition 5.2.39.

in the diagram

$$\begin{tikzcd}
\mathcal{F} \times_{B\mathrm{O}(R)} Y \arrow[rr, dashed] \arrow[ddd, dashed] & & \mathcal{F} \arrow[r, hook, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & \mathcal{E}\mathcal{X}^{\Delta[1]} \arrow[r, "\sim"] \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_1"] \arrow[dr, phantom, "\ulcorner"] & \mathcal{V}^{\hookrightarrow}(r,R)^{\Delta[1]} \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_1"] \\
& & B\mathrm{O}(r) \times B\mathrm{O}(R) \arrow[r, hook] \arrow[dd] & \mathcal{E}\mathcal{X}^{\times 2} \arrow[r, "\sim"] & \mathcal{V}^{\hookrightarrow}(r,R)^{\times 2} \\
& & & & \\
Y \arrow[rr, "F"] & & B\mathrm{O}(R) & &
\end{tikzcd},$$

$\mathcal{F}$ coincides with $B\mathrm{O}(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} B\mathrm{O}(R)$, and, by the pasting lemma and again by Theorem 3.3.1, also with

$$\begin{aligned}
\mathcal{E}\mathcal{X}^{\Delta[1]} \times_{\mathcal{E}\mathcal{X}^{\times 2}} (B\mathrm{O}(r) \times B\mathrm{O}(R)) &= B\mathrm{O}(r) \times_{\mathcal{E}\mathcal{X}^{\{0\}}} \mathcal{E}\mathcal{X}^{\Delta[1]} \times_{\mathcal{E}\mathcal{X}^{\{1\}}} B\mathrm{O}(R) \\
&\simeq (B\mathrm{O}(r) \times B\mathrm{O}(R-r)) \times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]}.
\end{aligned}$$

Thus, both sides in the statement are equivalent to $\mathcal{F} \times_{B\mathrm{O}(R)} Y$. □

In terms of lifts, Proposition 6.3.13 reads:

Corollary 6.3.14. *A solid F-structure, in the sense of Definition 6.3.3, on a smooth manifold X of dimension r, is a cartesian F-structure, in the sense of Definition 6.3.8, on the (trivially-linked) manifold X.*

6.4. Cartesian replacements, II: The linked case

We now generalise the discussion from cartesian structures on smooth spaces to those on linked spaces. Throughout this section, let $F\colon Y \to B\mathrm{O}(R) \subset \mathcal{V}^{\hookrightarrow}$ be a smooth tangential structure, and $\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N\right)$ be a linked manifold with $\dim M = n$, $\dim N = n+m$. For simplicity, let us assume $R = n+m$, leaving the obvious modifications for the case $R > n+m$ to the reader (cf. Remark 6.7.14). We will sometimes abuse notation by not distinguishing $B\mathrm{O}(n,m)$ from $\mathcal{E}\mathcal{X}(B\mathrm{O}(n,m))$, or $\mathfrak{S}$ from $\mathcal{E}\mathcal{X}(\mathfrak{S})$, etc., and also keep to Notation 6.1.3 throughout.

Since $\mathbf{U}\colon B\mathrm{O}(n,m) \hookrightarrow \mathcal{V}^{\hookrightarrow}$ is fully faithful, the (iterated) fibre product

$$\overline{Y}_{(n,m)} \coloneqq B\mathrm{O}(n,m) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y} = (\mathbf{U} \downarrow F)$$

as in Section 6.2 is simply a restriction, and so, in light of the results of Section 6.3, we are in a position to understand the meaning of a cartesian Y-structure in the linked setting.

Definition 6.4.1. A *cartesian Y-structure* on $\mathfrak{S}$ is a lift[8] of type

$$\begin{tikzcd}
& \overline{Y}_{(n,m)} \arrow[d] \\
\mathfrak{S} \arrow[r, "\mathrm{T}\mathfrak{S}"'] \arrow[ur, dotted, "t"] & B\mathrm{O}(n,m)
\end{tikzcd}.$$

[8]We consider honest lifts rather than homotopy lifts in view of Proposition 6.3.13.

The main goal of this section is to identify a span $\mathfrak{B}$ of spaces over $BO(n,m)$ such that span maps $\mathfrak{S} \to \mathfrak{B}$ that lift $T\mathfrak{S}$ induce cartesian Y-structures on $\mathfrak{S}$. This is achieved in Theorem 6.4.20 combined with Observation 6.4.19.

6.4.1. Low dimensions. At a point $p \in M$, let us write $\mathrm{T}_p \coloneqq \mathrm{T}\mathfrak{S}(p) \in BO(n)$. A point-wise lift

$$t_p \in BO(n,m) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y$$

is determined by a path of type

$$W_p \oplus \mathrm{T}_p \xrightarrow{t_p} F_p^M$$

in $BO(n+m)$ where

$$\begin{aligned} &F_p^M \in F(Y) \subset BO(n+m), \\ &W_p \in BO(m). \end{aligned}$$

Since

$$\mathrm{T}\mathfrak{S}|_M \colon M \hookrightarrow \mathfrak{S} \to BO(n,m)$$

factors through

$$M \xrightarrow{\mathrm{T}M} BO(n) \hookrightarrow BO(n,m)^{\sim} \hookrightarrow BO(n,m)$$

(note $BO(n,m)^{\sim} \simeq BO(n) \amalg BO(n+m)$ exactly like $\mathcal{V}^{\simeq} \simeq BO^{\infty}_{\amalg}$), $t|_M \colon M \to \overline{Y}_{(n,m)}$ factors through $BO(n) \times_{BO(n,m)} BO(n,m) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y} = \overline{Y}|_n$ (recall (6.3.9)), we identify W_-, via Proposition 6.3.13, as the 'normal bundle'

$$W \colon M \xrightarrow{t|_M} (BO(n) \times BO(m)) \times^{\mathrm{h}}_{BO(n+m)} Y \xrightarrow{\mathrm{pr}_2 \circ \mathrm{ev}_0} BO(m) \;.$$

of the induced solid F-structure $t|_M$ on M. Of course, $\mathrm{T}M$ can be similarly identified to be $\mathrm{pr}_1 \circ \mathrm{ev}_0 \circ t|_M$ and F^M_- to be $\mathrm{ev}_1 \circ t|_M$. Similarly, a lift t_q at $q \in N$ is a path of type $\mathrm{T}_q \xrightarrow{t_q} F^N_q$. That is, $t|_N$ factors, again as a special case of Proposition 6.3.13, through $BO(n+m)^{\Delta[1]} \times_{BO(n+m)^{\{1\}}} Y$ which projects to $BO(n+m)$ via ev_0 so that $\mathrm{T}N = \mathrm{ev}_0 \circ t|_N$.

Let us consider now a non-invertible exit 1-path $\gamma = (\widehat{\gamma}, 1)$ in $\mathcal{EX}(\mathfrak{S})$ and explicate the application of $\mathrm{T}\mathfrak{S}$ thereon: γ is determined by a path $\widehat{\gamma} \colon \widehat{p} \to q$ in N, and so $\mathrm{T}\mathfrak{S}(\gamma)$ will be determined by a path in $BO(n+m)_1$ with source $\oplus\left((\pi^*\mathrm{T}M \times \mathrm{N}_N M)\,(\widehat{p})\right) = \mathrm{N}_{\widehat{p}} \oplus \mathrm{T}_p M$, where $\mathrm{N} = \mathrm{N}_N M$, and actual path

$$\mathrm{T}_{\widehat{p}} N \xrightarrow{\mathrm{T}N(\widehat{\gamma})} \mathrm{T}_q N$$

which is consistent since $\mathcal{EX}(\mathfrak{S}) \to \mathcal{EX}(BO(n,m))$'s being induced by a span map $\mathfrak{S} \to BO(n,m)$ implies

$$\mathrm{N}_{\widehat{p}} \oplus \mathrm{T}_p M = \mathrm{T}_{\widehat{p}} N. \tag{6.4.2}$$

Let us place all players involved in a diagram:

$$\begin{tikzcd}
N_1 & BO(n+m)_1 \\
\Cup & \Cup \\
\left(\widehat{p} \xrightarrow{\widehat{\gamma}} q\right) \arrow[r, "\mathrm{d}\iota"] \arrow[d, dash] & \left(\mathrm{N}_{\widehat{p}} \oplus \mathrm{T}_p M \xrightarrow{\mathrm{T}_{\widehat{\gamma}} N} \mathrm{T}_q N\right) \arrow[d, dash] \\
\left(p \underset{\gamma}{\to} q\right) \arrow[r, "\mathrm{T}_\gamma \mathfrak{S}"'] & \left(\mathrm{T}_p M \underset{\mathrm{T}_L \mathfrak{S}(\gamma)}{\longrightarrow} \mathrm{T}_q N\right) \\
\Cap & \Cap \\
\mathcal{E}\mathcal{X}(\mathfrak{S})_1 & BO(n,m)_1
\end{tikzcd} \tag{6.4.3}$$

Now, a lift t_γ (or rather its underlying 2-path) necessarily factors as $t_\gamma \colon \Delta[1] \times \Delta[1] \to BO(n+m) \hookrightarrow \mathcal{V}^{\hookrightarrow}$ and, resuming the notation from the beginning of Section 5.2, we have

$$\left(p \xrightarrow{\gamma} q\right) \overset{t}{\mapsto} \left(\begin{tikzcd}
F_p^M \arrow[rr, "\rho"] & & F_q^N \\
& (H,\zeta) & \\
\mathrm{T}_p M \arrow[uu, "{(W_p, t_p)}"] \arrow[uurr, dashed] \arrow[rr, "{(\mathrm{N}_{\widehat{p}}, \mathrm{T}_{\widehat{\gamma}} N)}"'] & & \mathrm{T}_q N \arrow[uu, "{(0, t_q)}"']
\end{tikzcd}\right) \tag{6.4.4}$$

in view of (6.4.3). That $t_\gamma \in \left(\overline{Y}_{(n,m)}\right)_1$ means that the lower horizontal path is to be in $\mathcal{V}^{\hookrightarrow}|_{n,m}$ (recall Corollary 6.3.12) and the upper horizontal path in $F(Y)$. This implies that there is an induced concatenated path

$$W_p \to \mathrm{N}_{\widehat{p}}. \tag{6.4.5}$$

Indeed, the square is shorthand two 3-simplices in $\mathcal{B}^{\oplus}\mathrm{O}$ of the form

$$\begin{tikzcd}
& 2 \arrow[dr, "0"] & \\
1 \arrow[ur, "W_p"] \arrow[rr, "H"] & & 3 \\
& 0 \arrow[ul, "\mathrm{T}_p M"] \arrow[uu, "F_p M"] \arrow[ur, "F_q N"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 2 \arrow[dr, "0"] & \\
1 \arrow[ur, "\mathrm{N}_{\widehat{p}}"] \arrow[rr, "H"] & & 3 \\
& 0 \arrow[ul, "\mathrm{T}_p M"] \arrow[uu, "\mathrm{T}_q N"] \arrow[ur, "F_q N"'] &
\end{tikzcd}, \tag{6.4.6}$$

which correspond to the upper and lower triangle respectively, *such that* the 013-faces $d_2^{\mathcal{B}^{\oplus}\mathrm{O}}(t_\gamma)$, filled by ζ, coincide, for that is the diagonal/common hypotenuse. The path (6.4.5) is induced by the two top triangles filled by paths $W_p \to H$, $\mathrm{N}_{\widehat{p}} \to H$.

6.4.2. Sufficient material to construct t. Let $\ell \in L$ and see it, via the constant-loop inclusion

$$L \hookrightarrow P(L) \overset{\iota_*}{\hookrightarrow} P(N), \tag{6.4.7}$$

as an exit path

$$\pi(\ell) \to \iota(\ell) \quad \text{in} \quad \mathfrak{S}.$$

Taking $p = \pi(\ell)$ and $q = \iota(\ell)$, we have that the path $\mathrm{T}_{\widehat{\gamma}}N = \mathrm{T}_\ell N$ is the constant path $\mathrm{N}_\ell \oplus \mathrm{T}_p M \xrightarrow{\mathrm{id}} \mathrm{T}_q N$. In particular, along $L \hookrightarrow P(N)$ we may take $H = \mathrm{N}_{\widehat{p}}$ and so (6.4.5) is given without concatenation by the first 3-simplex in (6.4.6). In fact, let us for simplicity take

$$H = W_p = \mathrm{N}_{\widehat{p}} \tag{6.4.8}$$

and use the constant paths.

So it suffices to provide a filler for

$$t_\gamma = \left(\begin{tikzcd} F^M_p \arrow[rr, "\rho"] & & F^N_{\iota(\ell)} \\ & \mathrm{N}_\ell \oplus \mathrm{T}_p M \arrow[ul, "t_p"] \arrow[ur, "t_q"'] & \end{tikzcd} \right)$$

in order to construct (6.4.4) along $L \hookrightarrow P(L)$.

Remark 6.4.9. The constant-loop inclusion defines a map

$$L \to (M \downarrow N) = M \times_{\mathcal{EX}^{\{0\}}} \mathcal{EX}^{\Delta[1]} \times_{\mathcal{EX}^{\{1\}}} N$$

at vertex level by

$$\ell \mapsto (\pi(\ell), \mathrm{const}_\ell, \iota(\ell)) .$$

This map is in fact an equivalence (Proposition 6.6.7), realising the equivalence of Theorem 3.4.1 (cf. Observation 6.7.1).

Varying ℓ and rewriting, we see that this provides a point-wise filler for the triangle

$$\begin{tikzcd} \pi^* F^M \arrow[rr, "\rho"] \arrow[dr, dotted, bend right] & & \iota^* F^N \arrow[dl, dotted, bend left] \\ & \mathrm{N} \oplus \pi^* \mathrm{T}M \arrow[ul, "\pi^*(t|_M)"] \arrow[ur, "\iota^*(t|_N)"'] \arrow[d, dotted] & \\ & L & \end{tikzcd} \tag{6.4.10}$$

of bundle isomorphisms over L. Our simplification amounts then, along $L \hookrightarrow P(L)$, to saying $\pi^* W = \mathrm{N}_N M$ and so

$$\mathrm{N} \oplus \pi^* \mathrm{T}M = \pi^*(W \oplus \mathrm{T}M).$$

The bundles $F^{M/N}$ are $F^{M/N} \colon M/N \xrightarrow{t|_{M/N}} \overline{Y}_{(n,m)} \xrightarrow{\mathrm{ev}_1} Y \xrightarrow{F} B\mathrm{O}(n+m)$.

The two legs of (6.4.10) give two Y-structures on the bundle $\mathrm{N} \oplus \pi^* \mathrm{T}M \to L$, and ρ together with the filler 2-simplex, for which we will collectively write

ρ, is a path between them:

$$\left(\pi^*(t|_M) \xrightarrow{\rho} \iota^*(t|_N)\right) \quad \text{in} \quad Y\text{-red}\,(\mathrm{N} \oplus \pi^*\mathrm{T}M) \tag{6.4.11}$$

where the space

$$Y\text{-red}(E \to X) = \mathrm{Map}_{/B\mathrm{O}(n+m)}(E, Y)$$

is that of the lifts of $E\colon L \to B\mathrm{O}(n+m)$ to $F\colon Y \to B\mathrm{O}(n+m)$, for E a rank-$(n+m)$ bundle on X.

We may now relax the assumption (6.4.8) back in the following way.

Lemma 6.4.12. *Suppose*

$$\rho\colon L \to B\mathrm{O}(n+m)^{I\times I}$$

is a map that satisfies the following conditions:

(1) $\left(L \to B\mathrm{O}(n+m)^{I\times I} \xrightarrow{\mathrm{ev}_0^2} B\mathrm{O}(n+m)^{I\times\{0\}}\right) = \pi^* t_M,$

(2) $\left(L \to B\mathrm{O}(n+m)^{I\times I} \xrightarrow{\mathrm{ev}_1^2} B\mathrm{O}(n+m)^{I\times\{1\}}\right) = \iota^* t_N,$

(3) $L \to B\mathrm{O}(n+m)^{I\times I} \xrightarrow{\mathrm{ev}_1^1} B\mathrm{O}(n+m)^{\{1\}\times I}$ *hits* $\mathrm{Im}(F)$, *which is to say it lifts along* $Y^I \xrightarrow{F_*} B\mathrm{O}(n+m)^I$.

Then it yields a map

$$\rho\colon L \to (\pi^* t_M \downarrow \iota^* t_N)\,.$$

We will prove the preceding lemma after noting some of its consequences.

Sometimes we also write ρ in the form

$$L \to \mathrm{Hom}_{\overline{Y}_{(n,m)}}\left(\pi^* t_M(-), \iota^* t_N(-)\right)$$

for its suggestiveness, rather than the arrow notation for comma categories.[9]

Diagrammatically, we have

$$\begin{tikzcd} \iota^*\mathrm{T}N \arrow[r, "\iota^* t_N"] & \iota^*(F^N) \\ \pi^*(W \oplus \mathrm{T}M) \arrow[u] \arrow[r, "\pi^* t_M"] & \pi^*(F^M) \arrow[u, "\text{within } \mathrm{Im}(F)"'] \end{tikzcd} \,. \tag{6.4.13}$$

When evaluated at a point $\ell \in L$, this gives a diagram in $\mathcal{V}^{\hookrightarrow}$ via

$$B\mathrm{O}(n+m)^{I\times I} \hookrightarrow (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]}.$$

If (6.4.8) holds, we may take the left vertical path to be constant at $\pi^*(W \oplus \mathrm{T}M) = \mathrm{N} \oplus \pi^*\mathrm{T}M = \iota^*\mathrm{T}N$, or otherwise we can induce it directly by the path η classifying the isomorphism $\pi^* W \cong \mathrm{N}$ as $\eta \oplus \mathrm{id}_{B\mathrm{O}(n)}$. If the former, then ρ is of type

$$\rho\colon L \to B\mathrm{O}(n+m)^{\Delta^2},$$

[9]This will be natural for readers familiar with dependent pair types, a.k.a. Σ-types (see e.g. [77, §1.6] for an exposition).

as was implicit in (6.4.11). Indeed, this incarnation is also relevant for our purposes, so let us state the corresponding version of the Lemma.

The following definition is extracted from (6.4.10).

Notation 6.4.14. We denote the (ordinary) limit of the diagram

$$\begin{tikzcd}
 & BO(n+m)^{\Delta^2} \arrow[dl, two heads] \arrow[dr, two heads] & \\
BO(n+m)^{\{0\}} & & BO(n+m)^{\Delta^{\{1<2\}}} \\
BO(n)\times BO(m) \arrow[u, hook, "\oplus"] & & Y^I \arrow[u, "F_*"]
\end{tikzcd}$$

of spaces by $BO(n+m)^{\Delta^2}|^{\oplus}_Y$.

Lemma 6.4.15. *Suppose*

$$\rho\colon L \to BO(n+m)^{\Delta^2}|^{\oplus}_Y$$

is a map such that its composition with

$$BO(n+m)^{\Delta^2} \to BO(n+m)^{\Delta^{\{0<1\}}}$$

is $\pi^* t_M$*, and its composition with*

$$BO(n+m)^{\Delta^2} \to BO(n+m)^{\Delta^{\{0<2\}}}$$

is $\iota^* t_N$*. Then it yields a map*

$$\rho\colon L \to \mathrm{Hom}_{\overline{Y}_{(n,m)}}\left(\pi^* t_M(-), \iota^* t_N(-)\right).$$

PROOF. Push Lemma 6.4.12 along $I \times I \simeq \Delta^2$. □

Remark 6.4.16. The simplicity of the statement of Lemma 6.4.15 is due to our running assumption that $R = n + m$, i.e., that the rank of the smooth structure coincides with the dimension of N. We leave the modification of the definition of $BO(n+m)^{\Delta^2}|^{\oplus}_Y$ (and of the proof) for the general case $R \geq n+m$ to the reader.

In any case, the left vertical path in (6.4.13) need not be provided separately when attempting to construct a cartesian structure t. If $Y = *$ (framings of rank $n+m$), the right vertical path can be collapsed as well, and we would have a globular 2-morphism, but this is a very special case.

Definition 6.4.17. We call a ρ as in Lemma 6.4.12 or Lemma 6.4.15 a *compatibility* between the structures t_M, t_N. If it exists, we call the structures on M and N *compatible* over L.

PROOF OF LEMMA 6.4.12. Such a map yields

$$\begin{aligned}
\rho\colon L &\to (L, \pi^* t_M) \times_{\overline{Y}^{\{0\}}_{(n,m)}} \overline{Y}^{\Delta[1]}_{(n,m)} \times_{\overline{Y}^{\{1\}}_{(n,m)}} (L, \iota^* t_N) = (\pi^* t_M \downarrow \iota^* t_N) \\
&\hookrightarrow (L, \pi^* t_M) \times_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\{1\}}} (L, \iota^* t_N)
\end{aligned}$$

along the inclusion $\overline{Y}_{(n,m)} \hookrightarrow (\mathcal{V}^{\hookrightarrow})^{\Delta[1]}$. Indeed, that

$$\rho\colon L \to BO(n+m)^{I\times I} \hookrightarrow (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]}$$

descends to

$$\rho\colon L \to \overline{Y}^{\Delta[1]}_{(n,m)} \cong BO(n,m)^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}\times\Delta[1]}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}\times\Delta[1]}} Y^I$$

is clear: that

$$\rho\colon L \to (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]} \to (\mathcal{V}^{\hookrightarrow})^{\{0\}\times\Delta[1]}$$

factors through $BO(n,m)^{\Delta[1]}$ is trivial since it already factors through $BO(n+m)^I \hookrightarrow (BO(n,m)^{\sim})^{\Delta[1]} \simeq BO(n)^I \amalg BO(n+m)^I$, and that

$$\rho\colon L \to (\mathcal{V}^{\hookrightarrow})^{\Delta[1]\times\Delta[1]} \to (\mathcal{V}^{\hookrightarrow})^{\{1\}\times\Delta[1]}$$

factors through Y^I is imposed by Condition 3.

Further, that $\rho\colon L \to \overline{Y}^{\Delta[1]}_{(n,m)}$ evaluates to $\pi^* t_M$ resp. $\iota^* t_N$ under ev_0 resp. ev_1 is imposed by Condition 1 resp. 2, so that the two legs $L \to L$ in the map to the iterated fibre product can be given by the identity. □

At a point $\ell \in L$ with $\pi(\ell) = p$ and $\iota(\ell) = q$, we have

$$\begin{aligned} \rho_\ell &\in \{\pi^* t_M(\ell)\} \times_{\overline{Y}^{\{0\}}_{(n,m)}} \overline{Y}^{\Delta[1]}_{(n,m)} \times_{\overline{Y}^{\{1\}}_{(n,m)}} \{\iota^* t_N(\ell)\} \\ &= \{t_M(p)\} \times_{\overline{Y}^{\{0\}}_{(n,m)}} \overline{Y}^{\Delta[1]}_{(n,m)} \times_{\overline{Y}^{\{1\}}_{(n,m)}} \{t_N(q)\} \\ &= \mathrm{Hom}_{\overline{Y}_{(n,m)}}(t_M(p), t_N(q)). \end{aligned}$$

Definition 6.4.18. A *solid Y-structure* on $\mathfrak{S}$ consists of

- a Y-structure on N determined by a bundle isomorphism, $\mathrm{T}N \overset{t_N}{\cong} F^N$
- a solid Y-structure on M determined by a bundle isomorphism $W \oplus \mathrm{T}M \overset{t_M}{\cong} F^M$ such that $\pi^* W \cong \mathrm{N}_N M$,
- and a compatibility $\rho\colon L \to \mathrm{Hom}_{\overline{Y}_{(n,m)}}(\pi^* t_M(-), \iota^* t_N(-))$.

Observation 6.4.19. *A solid Y-structure on $\mathfrak{S}$ is a lift of $\mathrm{T}\mathfrak{S}$ to the following span over $BO(n,m)$:*

$$\begin{tikzcd}
 & & & BO(n+m)^{\Delta^2}|^{\oplus}_{Y} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd] & \\
 & & BO(n+m)^{\Delta^{\{0<1\}}}|^{\oplus}_{Y} \arrow[ddd] & & BO(n+m)^{\Delta^{\{0<2\}}}|^{\oplus}_{Y} \arrow[dd] \\
L \arrow[rrr, "\mathrm{T}_L\mathfrak{S}"] \arrow[dr, hook, "\iota"] \arrow[dd, two heads, "\pi"'] & & & BO(n) \times BO(m) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, hook, "\oplus"] & \\
 & N \arrow[rrr, "\mathrm{T}N"'] & & & BO(n+m) \\
M \arrow[rr, "\mathrm{T}M"] & & BO(n) & &
\end{tikzcd}$$

Proof. Note that

$$BO(n+m)^{\{0<1\}}|_Y^{\oplus} \simeq \overline{Y}|_n$$

by Proposition 6.3.13, so a lift of TM to it is precisely a solid Y-structure on M. Similarly,

$$BO(n+m)^{\Delta^{\{0<2\}}}|_Y^{\oplus} \simeq BO(n+m)^I \times_{BO(n+m)^{\{1\}}} Y \simeq \overline{Y}|_{n+m},$$

so a lift of TN is a Y-structure on N. The statement now follows from Lemma 6.4.15. □

The following is the main result of the present chapter.

Theorem 6.4.20. *The linked manifold $\mathfrak{S}$ possesses a solid Y-structure if and only if it possesses a cartesian Y-structure.*

The proof of the two directions are split into Sections 6.6 and 6.7. However, we will first discuss the 'reason' why one of the directions is true, and in doing so highlight the technical difficulties in obtaining a full proof of the statement, which consists in various contractible choices that need to be made consistently. The goal of the *box construction* of Section 6.6 is to organise them, in essence, to a single contractible choice.

6.5. The *only-if* statement in Theorem 6.4.20 to second order

The content of the proof of the *only-if* statement in terms of the corresponding topological categories is as follows. Of course, $t_M \amalg t_N$ defines the restriction of t to $\mathfrak{S}^\sim = M \amalg N$. Let now $p \in M$, $q \in N$. We can provide t on morphisms from p to q, i.e., a map

$$P_{L_p,q} \to \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_p, t_q) = \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q)).$$

The compatibility

$$\rho\colon L \to \mathrm{Hom}_{\overline{Y}_{(n,m)}}\left(\pi^* t_M(-), \iota^* t_N(-)\right), \tag{6.5.1}$$

gives by restriction a map

$$\rho|_p\colon P_{L_p,q} \xrightarrow{\mathrm{ev}_0} L_p \xrightarrow{\rho|} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(\mathrm{ev}_0(-))), \tag{6.5.2}$$

where $\mathrm{ev}_0(-)$ in the second argument takes $\widehat{\gamma} \in P_{L_p,q}$ to its initial point.[10] Similarly, the restriction of t_N by restriction gives a map

$$t_N|\colon P_{L_p,q} \to \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(\mathrm{ev}_0(-)), t_N(q)). \tag{6.5.3}$$

We may now 'compose' to obtain the desired map

$$\begin{aligned} P_{L_p,q} &\xrightarrow{\rho| \times t_N|} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(\mathrm{ev}_0(-))) \times \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(\mathrm{ev}_0(-)), t_N(q)) \\ &\xrightarrow{\circ} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q)). \end{aligned}$$

This gives the statement to first order.

Notation 6.5.4. From now on, for the sake of readability, we will sometimes write $[-,-]_{(-)} \coloneqq \mathrm{Hom}_{(-)}(-,-)$.

[10] As usual, we do not distinguish L from $\iota(L)$ in notation.

For the second order, we will check the functoriality of this map to first order directly:

$$\begin{tikzcd}
{[p,p']_{\mathcal{EX}} \times [p',q]_{\mathcal{EX}}} \arrow[rr] \arrow[d, dashed, "\sim"] & & {[p,q]_{\mathcal{EX}}} \arrow[d, dashed, "\sim"] \\
P(M)_{p,p'} \times P(N)_{L_{p'},q} \arrow[rr] \arrow[d] & & P(N)_{L_p,q} \arrow[d] \\
{[t_M(p), t_M(p')]_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}} \times [t_M(p'), t_N(q)]_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}} \arrow[rr] & & {[t_M(p), t_N(q)]_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}}
\end{tikzcd} \tag{6.5.5}$$

homotopy-commutes, as does

$$\begin{tikzcd}
{[p,q] \times [q,q']} \arrow[r] \arrow[d, dashed, "\sim"] & {[p,q']} \arrow[d, dashed, "\sim"] \\
P(N)_{L_p,q} \times P(N)_{q,q'} \arrow[r] \arrow[d] & P(N)_{L_p,q'} \arrow[d] \\
{[t_M(p), t_N(q)] \times [t_N(q), t_N(q')]} \arrow[r] & {[t_M(p), t_N(q')]}
\end{tikzcd} \tag{6.5.6}$$

where $p \in M$ and $q, q' \in N$.

Let us consider (6.5.5) first. Unpacking the construction, the lower square is

$$\begin{aligned}
P(M)_{p,p'} \times P(N)_{L_{p'},q} &\xrightarrow{t_M \times (\rho|_{p'} \times t_N|)} \\
&[t_M(p), t_M(p')] \times ([t_M(p'), t_N(\mathrm{ev}_0(-))] \times [t_N(\mathrm{ev}_0(-)), t_N(q)]) \\
&\xrightarrow{\mathrm{id} \times \circ} [t_M(p), t_M(p')] \times [t_M(p'), t_N(q)] \\
&\xrightarrow{\circ} [t_M(p), t_N(q)]
\end{aligned}$$

counter-clockwise, and

$$\begin{aligned}
P(M)_{p,p'} \times P(N)_{L_{p'},q} \to P(N)_{L_p,q}& \\
&\xrightarrow{\rho|_p \times t_N|} [t_M(p), t_N(\mathrm{ev}_0(-))] \times [t_N(\mathrm{ev}_0(-)), t_N(q)] \\
&\xrightarrow{\circ} [t_M(p), t_N(q)]
\end{aligned}$$

clockwise. Let now $\delta \in P(M)_{p,p'}$, $\widetilde{\gamma} \in P(N)_{L_{p'},q}$, and let $\widetilde{\epsilon} \in P(N)_{L_p,q}$ a choice of composition, with composition 2-path $\Gamma = (\widetilde{\Gamma}, 2) \in \mathcal{P}_1^{\Delta} \subset \mathcal{EX}_2$ necessarily of maximal exit index. This is to say that

$$\begin{aligned}
d_2\Gamma &= \pi(d_2\widetilde{\Gamma}) = \delta, \\
d_0\Gamma &= (d_0\widetilde{\Gamma}, \flat_{2,0}) = (d_0\widehat{\Gamma}, 1) = (\widetilde{\epsilon}, 1), \\
d_1\Gamma &= (d_0\widetilde{\Gamma}, \flat_{2,1}) = (\widetilde{\gamma}, 1)
\end{aligned}$$

where $\widetilde{\delta} \coloneqq d_2\widetilde{\Gamma}$ is a path in L covering δ. Say $\ell \in L_p$ is the source of $\widehat{\delta}$ and $\ell' \in L_{p'}$ its target, so that the two compositions in the diagram evaluate to

$$\begin{aligned}(\delta, \widetilde{\gamma}) &\mapsto \left(t_M(p) \xrightarrow{t_M(\delta)} t_M(p'), t_M(p') \xrightarrow{\rho(\ell')} t_N(\ell'), t_N(\ell') \xrightarrow{t_N(\widetilde{\gamma})} t_N(q)\right)\\ &\mapsto \left(t_M(p) \xrightarrow{t_M(\delta)} t_M(p'), t_M(p') \xrightarrow{\rho(\ell')*t_N(\widetilde{\gamma})} t_N(q)\right)\\ &\mapsto \left(t_M(p) \xrightarrow{t_M(\delta)*\rho(\ell')*t_N(\widetilde{\gamma})} t_N(q)\right)\end{aligned}$$

counter-clockwise, and to

$$\begin{aligned}(\delta, \widetilde{\gamma}) \mapsto \widetilde{\epsilon} &\mapsto \left(t_M(p) \xrightarrow{\rho(\ell)} t_N(\ell), t_N(\ell) \xrightarrow{t_N(\widetilde{\epsilon})} t_N(q)\right)\\ &\mapsto \left(t_M(p) \xrightarrow{\rho(\ell)*t_N(\widetilde{\epsilon})} t_N(q)\right)\end{aligned}$$

clockwise, but Γ provides two fillers, namely ρ applied to $\widetilde{\delta}$ which underlies its low edge, and t_N applied to $\widetilde{\Gamma}$ itself, that fill

$$\begin{tikzcd} t_M(p) \arrow[rr, "t_M(\delta)"] \arrow[d, "\rho(\ell)"'] & & t_M(p') \arrow[d, "\rho(\ell')"] \\ & \rho(\widetilde{\delta}) & \\ t_N(\ell) \arrow[rr, "t_N(\widetilde{\delta})" description] \arrow[dr, "t_N(\widetilde{\epsilon})"'] & t_N(\widetilde{\Gamma}) & t_N(\ell') \arrow[dl, "t_N(\widetilde{\gamma})"] \\ & t_N(q) & \end{tikzcd} \tag{6.5.7}$$

as depicted. This provides exactly the desired homotopy $t_M(\delta)*\rho(\ell')*t_N(\widetilde{\gamma}) \sim \rho(\ell) * t_N(\widetilde{\epsilon})$.

As for (6.5.6), let $\widetilde{\delta} \in P(N)_{L_p,q}$, $\gamma \in P(N)_{q,q'}$ and let $\widetilde{\epsilon} \in P(N)_{L_p,q'}$ be a choice of composition with filler $\Gamma = (\widetilde{\Gamma}, q) \in \mathcal{EX}_1$ necessarily of minimal exit index satisfying the index-1 analogue of the above simplicial relations.

Say now $\ell \in L_p$ is the source of $\widetilde{\delta}$ and of $\widetilde{\varepsilon}$. The compositions in the lower square evaluate to

$$\begin{tikzcd} (\widetilde{\delta}, \gamma) \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{\varepsilon} \arrow[d, mapsto] \\ \left(t_M(p) \xrightarrow{\rho(\ell)*t_N(\widetilde{\delta})*t_N(\gamma)} t_N(q')\right) & \left(t_M(p) \xrightarrow{\rho(\ell)*t_N(\widetilde{\epsilon})} t_N(q')\right) \end{tikzcd}$$

but now it suffices to observe that the filler

$$t_N(\widetilde{\Gamma}) = \left(\begin{tikzcd} t_N(q) \arrow[r, "t_N(\gamma)"] & t_N(q') \\ t_N(\ell) \arrow[u, "t_N(\widetilde{\delta})"] \arrow[ur, "t_N(\widetilde{\epsilon})"'] & \end{tikzcd} \right)$$

induces the desired homotopy $\rho(\ell) * t_N(\widetilde{\delta}) * t_N(\gamma) \sim \rho(\ell) * t_N(\widetilde{\epsilon})$.

The problem with this method is that, while the map

$$P_{L_{p,q}} \to [t_M(p), t_N(q)]$$

on the morphism spaces is easy to specify, and its 'functoriality' to first order in the sense above is straightforward albeit notationally tedious, it is not clear how to translate these data to an actual ∞-functor $t\colon \mathcal{EX} \to \overline{Y}_{(n,m)}$. The technical issue is that even though the so-called Dwyer–Kan models $\mathrm{Hom}_{\mathcal{C}}^{(R/L)}(p,q)$ of morphism spaces in an ∞-category due to Joyal/Lurie are well-known to be weakly equivalent to the simplicial morphism spaces $\mathfrak{C}\mathcal{C}(p,q)$ in the rigidification $\mathfrak{C}\mathcal{C}$ ([51, 28, 27]), it is not clear how exactly to provide an honest simplicial functor $\mathfrak{C}\mathcal{EX} \to \mathfrak{C}\overline{Y}_{(n,m)}$ given all the choices (even if contractible) above. It also seems that the description of $\mathfrak{C}\mathcal{C}(p,q)$ in terms of necklaces due to Dugger–Spivak does not simplify matters in this particular case.

In any case, this approach would be technically somewhat unnatural. After rigidification, one obtains simplicial morphism spaces that are in general not even quasi let alone Kan, but rather have different combinatorial behaviour. For instance, Riehl showed in [65] that while all of their inner 2-horns admit fillers, some contain 3-horns without fillers, and all are 3-coskeletal, which is to say that spheres $\Delta/\partial\Delta$ of dimension at least 3 admit fillers.

Nevertheless, Lurie proved in [51], as did Dugger and Spivak in [27] using different methods, that the simplicial morphism spaces and the Kan morphism spaces are weakly equivalent, that is, weakly equivalent in the usual sense after taking geometric realisations. This applies in particular to Lurie's pinched morphism spaces, one variant of which was used in the proof of Theorem 3.3.1 – these are collectively termed the 'Dwyer–Kan models' in ibid.

In concrete terms, the contractible spaces of choices mentioned above do not have obviously specifiable elements that do the job, unlike our discussion in Section 5.2 in the context of the construction of the unpacking map.

6.6. The box construction

In order to make the idea of Section 6.5 a reality, we will introduce and work with variants of $\mathcal{EX}$, $\mathcal{V}^{\hookrightarrow}$ and $\overline{Y}_{(n,m)}$. As usual, we will sometimes not distinguish between L and $\iota(L)$.

Consider an exit 2-path $(\Gamma, 2) \in P_1^{\Delta} \subset \mathcal{E}\mathcal{X}_2$ of index 2:

$$\begin{tikzcd} & q \\ p \arrow[r] \arrow[ur] & p' \arrow[u] \end{tikzcd}$$

where, by construction, $p, p' \in M$, $q \in N$, and the notation for the edge $p \to p'$ includes a lift[11]

$$\mathbf{b}\Gamma_2 \coloneqq (\widehat{p} \to \widehat{p'}) \in L_1$$

up to L whose image under ι sits in the underlying 2-simplex $\Gamma \in N_2$, which is

$$\begin{tikzcd} & q \\ \iota(\widehat{p}) \arrow[r, "\iota(\mathbf{b}\Gamma_2)"'] \arrow[ur] & \iota(\widehat{p'}). \arrow[u] \end{tikzcd}$$

This path $\widehat{p} \to \widehat{p'}$ itself gives a square

$$\boxed{\mathbf{b}\Gamma_2} \colon \Delta[1] \times \Delta[1] \to \mathcal{E}\mathcal{X} \tag{6.6.1}$$

of the form

$$\begin{tikzcd} \iota(\widehat{p}) \arrow[r] & \iota(\widehat{p'}) \\ p \arrow[u, "\widehat{p}"] \arrow[r] & p' \arrow[u, "\widehat{p'}"] \end{tikzcd}$$

where we did not distinguish in notation between the points $\widehat{p}, \widehat{p'} \in L$ and their images under the constant loop inclusion $L \hookrightarrow P(L) \hookrightarrow P(N)$. The common diagonal $p \to \iota(\widehat{p'})$ is canonically provided by the upper horizontal path $\iota(\widehat{p} \to \widehat{p'})$ itself, seen as an exit 1-path $(\iota(\widehat{p} \to \widehat{p'}), 1)$ with source $\pi(\widehat{p}) = p$. In this way, the left triangle

$$\begin{tikzcd} \iota(\widehat{p}) \arrow[r, "\iota(\mathbf{b}\Gamma_2)"] & \iota(\widehat{p'}) \\ p \arrow[u, "\widehat{p}"] \arrow[ur, "(\iota(\mathbf{b}\Gamma_2), 1)"'] & \end{tikzcd} \tag{6.6.2}$$

in the square is filled by $(s_0\iota(\mathbf{b}\Gamma_2), 1)$.

To see this, note first that the exit index being 1 implies that the 1- and 2-faces are vertical, while the 0-face is upper, as desired. Since $d_i s_j = \mathrm{id}$ for $i \in \{j, j+1\}$, we have

$$d_1(s_0\iota(\mathbf{b}\Gamma_2), 1) = (d_1 s_0 \iota(\mathbf{b}\Gamma_2), \flat_{1,1}) = (\iota(\mathbf{b}\Gamma_2), 1)$$

and

$$d_0(s_0\iota(\mathbf{b}\Gamma_2), 1) = d_0 s_0 \iota(\mathbf{b}\Gamma_2) = \iota(\mathbf{b}\Gamma_2),$$

also as desired. Finally, because $d_i s_j = s_j d_{i-1}$ for $i > j+1$, we have

$$d_2(s_0\iota(\mathbf{b}\Gamma_2), 1) = (d_2 s_0 \iota(\mathbf{b}\Gamma_2), \flat_{1,2}) = (s_0 d_1 \iota(\mathbf{b}\Gamma_2), 1) = (s_0(\iota(\widehat{p})), 1),$$

[11]The prefix '$\mathbf{b}$' stands for 'base'.

which is precisely the constant loop at $\iota(\widehat{p})$ seen as an exit path (necessarily of index 1), for which we've been writing simply $\widehat{p}$.

Analogously, the right triangle

$$\begin{tikzcd}
 & \iota(\widehat{p'}) \\
p \arrow[ur, "{(\iota(\mathbf{b}\Gamma_2),1)}"] \arrow[r, "\pi\mathbf{b}\Gamma_2"'] & p' \arrow[u, "\widehat{p'}"']
\end{tikzcd} \tag{6.6.3}$$

in the square is filled by $(s_1\iota(\mathbf{b}\Gamma_2), 2)$. Indeed, the index implies that the 0- and 1-faces are vertical while the 2-face is low, as desired. Since $d_i s_j = s_{j-1} d_i$ for $i < j$, we have

$$d_0(s_1\iota(\mathbf{b}\Gamma_2), 2) = (d_0 s_1\iota(\mathbf{b}\Gamma_2), \flat_{2,0}) = (s_0 d_0 \iota(\mathbf{b}\Gamma_2), 1) = (s_0\iota(\widehat{p'}), 1)$$

which is, as discussed above, exactly what we denoted by $\widehat{p'}$. Next,

$$d_1(s_1\iota(\mathbf{b}\Gamma_2), 2) = (\iota(\mathbf{b}\Gamma_2), \flat_{2,1}) = (\iota(\mathbf{b}\Gamma_2), 1),$$

and finally

$$d_2(s_1\iota(\mathbf{b}\Gamma_2)) = \pi\mathbf{b}\Gamma_2.$$

This finishes the check. We can now glue, in $\mathcal{EX}$, the square $\boxed{\mathbf{b}\Gamma_2}$ with the underlying $\Gamma \in N_2$ along $\iota(\mathbf{b}\Gamma_2)$:

$$\begin{tikzcd}
 & q \\
\iota(\widehat{p}) \arrow[ur] \arrow[r, "\iota(\mathbf{b}\Gamma_2)"] & \iota(\widehat{p'}). \arrow[u] \\
p \arrow[u, "\widehat{p}"] \arrow[ur, "{(\iota(\mathbf{b}\Gamma_2),1)}" description] \arrow[r, "\pi\mathbf{b}\Gamma_2"'] & p' \arrow[u, "\widehat{p'}"']
\end{tikzcd} \tag{6.6.4}$$

Let us now consider an exit 2-path $(\Gamma, 1) \in \mathcal{P}_1^\Delta \subset \mathcal{EX}_2$ of index 1:

$$\begin{tikzcd}
q \arrow[r] & q' \\
p \arrow[u] \arrow[ur] &
\end{tikzcd}$$

with $p \in M$, $q, q' \in N$ where a lift $\widehat{p} \in L$ of p is implicit. Indeed, the base is $\mathbf{b}\Gamma_1 = \widehat{p} \in L_0$. It gives a 'square'

$$\boxed{\mathbf{b}\Gamma_1} \colon \Delta[1] \times \Delta\{0\} \to \mathcal{EX} \tag{6.6.5}$$

which is simply

$$\begin{tikzcd}
\iota(\widehat{p}) \\
p. \arrow[u, "\widehat{p}", "=s_0\iota(\widehat{p})"']
\end{tikzcd}$$

Gluing with the underlying $\Gamma \in N_2$ gives now

$$\begin{tikzcd} q \arrow[r] & q' \\ \iota(\widehat{p}) \arrow[u] \arrow[ur] & \\ p. \arrow[u, "\widehat{p}", "=s_0\iota(\widehat{p})"'] & \end{tikzcd} \tag{6.6.6}$$

The following should be thought of as the 'constant exit loop inclusion' in full generality.

Proposition 6.6.7. *The assignments* (6.6.5) *and* (6.6.1) *extend to an* ∞-*functor*

$$\square\colon L \to \mathcal{E}\mathcal{X}^{\Delta[1]}.$$

Moreover, $\square$ *factors through* $(M \downarrow N) \hookrightarrow \mathcal{E}\mathcal{X}^{\Delta[1]}$.

Proof. Let $n \geq 0$ and $\gamma \in L_n$. We define its upper part, the restriction of

$$\boxed{\gamma}\colon \Delta[1] \times \Delta[n] \to \mathcal{E}\mathcal{X}$$

along $\Delta[n] \simeq \Delta\{1\} \times \Delta[n] \hookrightarrow \Delta[1] \times \Delta[n]$, to be[12]

$$\boxed{\gamma}_{\Delta\{1\}\times\Delta[n]} := \iota(\gamma) \in M_n \subset \mathcal{E}\mathcal{X}_n, \tag{6.6.8}$$

and, its low part, the restriction along $\Delta[n] \simeq \Delta\{0\} \times \Delta[n] \hookrightarrow \Delta[1] \times \Delta[n]$ to be

$$\boxed{\gamma}_{\Delta\{0\}\times\Delta[n]} := \pi(\gamma) \in N_n \subset \mathcal{E}\mathcal{X}_n. \tag{6.6.9}$$

Further, given $i \in [n]$, we define the restriction along $\Delta[1] \simeq \Delta[1] \times \Delta\{i\} \hookrightarrow \Delta[1] \times \Delta[n]$ to be

$$\boxed{\gamma}_{\Delta[1]\times\Delta\{i\}} := (s_0\iota(\gamma|_i), 1) = (s_0\iota((\{i\} \hookrightarrow [n])^*\gamma), 1) \in \mathcal{P}_0^\Delta \subset \mathcal{E}\mathcal{X}_1.$$

This is consistent: the identification $\Delta[1] \simeq \Delta[1] \times \Delta\{i\}$ prescribes that the restriction of $(s_0\iota(\gamma|_i), 1)$ along $\Delta\{0\} \times \Delta\{i\} \hookrightarrow \Delta[1] \times \Delta\{i\}$ is simply

$$d_1((s_0\iota(\gamma|_i), 1)) = \pi(d_1 s_0 \gamma|_i) = \pi(\gamma|_i),$$

as desired – similarly for the upper part:

$$d_0((s_0\iota(\gamma|_i), 1)) = \iota(d_0 s_0 \gamma|_i) = \iota(\gamma|_i).$$

This reproduces (6.6.5) and (6.6.1).

Let now $j \in \{1, \ldots, n+1\}$ and $\mathcal{S}_j\colon \Delta[n+1] \hookrightarrow \Delta[1] \times \Delta[n]$ be the exit shuffle of index j. We define the restriction along $\mathcal{S}_j$ to be

$$\boxed{\gamma}_{\mathcal{S}_j} := (s_{j-1}\iota(\gamma), j)\,.$$

[12]We abuse notation very slightly by forgetting the identification $\Delta[n] \simeq \Delta\{1\} \times \Delta[n]$. We will resume this type of abuse below.

This systematises our filling of the triangles (6.6.2) and (6.6.3). Let us check that it is consistent with the definitions of $\boxed{\gamma}_{\Delta\{0/1\}\times\Delta[n]}$ and $\boxed{\gamma}_{\Delta[1]\times\Delta\{i\}}$ we have given.

For the low part, we must have that the low part[13] of $\boxed{\gamma}_{\mathcal{S}_j}$ coincides with the appropriate face of $\boxed{\gamma}_{\Delta\{0\}\times\Delta[n]}$. The low part is the restriction along the identity inclusion $\Delta[j-1] \hookrightarrow \Delta[n+1]$. Since $(\Delta[j-1] \hookrightarrow \Delta[n+1])^* s_{j-1} = (\sigma_{j-1} \circ ([j-1] \hookrightarrow [n+1]))^*$ and since $[j-1] \hookrightarrow [n+1] \xrightarrow{\sigma_{j-1}} [n]$ coincides with the identity inclusion $[j-1] \hookrightarrow [n]$, we have that

$$(\Delta[j-1] \overset{\mathrm{id}}{\hookrightarrow} \Delta[n+1])^*(s_{j-1}\iota(\gamma), j) = \pi(\gamma|_{0,\dots,j-1}),$$

as desired. (Alternatively, use the simplicial identities and the formula for $\flat_{-,-}$ iteratively.)

On the other hand, the upper part[14] is the restriction along $\Delta[n+1-j] = \Delta\{j,\dots,n+1\} \hookrightarrow \Delta[n+1]$. Since $\{j,\dots,n+1\} \hookrightarrow [n+1] \xrightarrow{\sigma_{j-1}} [n]$ is $\Delta[n+1-j] = \Delta\{j-1,\dots,n\} \hookrightarrow \Delta[n]$, we have

$$(\Delta\{j,\dots,n+1\} \hookrightarrow \Delta[n+1])^*(s_{j-1}\iota(\gamma), j) = \iota(\gamma|_{j-1,\dots,n}).$$

This is as desired, since $\mathcal{S}_j(k) = (1, i-1)$ for $k \geq j$, so that the upper part as picked out in the simplex category $\mathbf{\Delta}$ by the image of $[n+1-j] \overset{+j}{\hookrightarrow} [n+1] \overset{\mathcal{S}_j}{\hookrightarrow} [1]\times[n]$, which is $\{1\}\times\{j-1,\dots,n\}$, is precisely $\boxed{\gamma}_{\{1\}\times\{j-1,\dots,n\}} = \iota(\gamma|_{j-1,\dots,n})$.

Finally, we must show that the restrictions along the $\mathcal{S}_j$, thus defined, glue. We will then have defined $\boxed{\gamma}$ on every non-degenerate $(n+1)$-simplex of $\Delta[1]\times\Delta[n]$ consistently, which defines it on their colimit, which coincides with the colimit $\Delta[1]\times\Delta[n] \simeq \mathrm{colim}_{\Delta[i]\to\Delta[1]\times\Delta[n]}\Delta[i]$ itself, so that $\boxed{\gamma}$ will be defined.

Let therefore a pair of distinct exit indices $j < j'$ in $\{1,\dots,n+1\}$ be given. The intersection of the images of $\mathcal{S}_j, \mathcal{S}_{j'} \colon [n+1] \hookrightarrow [1]\times[n]$ as picked out within $\mathbf{\Delta}$ consists of a purely low part, $\{0\}\times[j-1]$, and a purely upper part, $\{1\}\times\{j'-1,\dots,n+1\}$. Let us write $\delta \coloneqq j'-j$ and consider the map

$$\mathcal{S}_{j\cap j'} \colon \Delta[n+1-\delta] \hookrightarrow \Delta[1]\times\Delta[n]$$

induced by the map $\mathcal{S}_{j\cap j'} \colon [n+1-\delta] \hookrightarrow [1]\times[n]$ given by

$$i \mapsto \begin{cases} (0,i), & i < j \\ (1, i-1+\delta), & i \geq j. \end{cases}$$

In other words, $\mathcal{S}_{j\cap j'}$ is like $\mathcal{S}_j$ except with upper part shifted by $j'-j$, and so that (the images in $[1]\times[n]$ of) its low and upper parts coincide precisely with those of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$, respectively. We have that $\mathcal{S}_{j\cap j'}$ picks out precisely the intersection of the images of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ within $\Delta[1]\times\Delta[n]$. Moreover, its factorisation through $\mathcal{S}_j$ as well as through $\mathcal{S}_{j'}$ is given by identifying

$$\Delta[n+1-\delta] = \Delta\{0,\dots,j-1,j',\dots,n+1\}$$

[13]This means of course the sub-simplicial set generated by the low vertices of the exit path.
[14]generated by the upper vertices

in the sense that

$$\begin{tikzcd}
\Delta\{0,\dots,j-1,j',\dots,n+1\} \arrow[r, hook, "\mathcal{S}_{j\cap j'}"] \arrow[d, hook, "\Theta"'] & \Delta[1]\times\Delta[n] \\
\Delta[n+1] \arrow[ur, hook, "\mathcal{S}_j \text{ or } \mathcal{S}_{j'}"'] &
\end{tikzcd}$$

commutes. This can be checked within $\mathbf{\Delta}$: since $j < j'$, the restrictions of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ along $[0,\dots,j-1] \hookrightarrow [n+1]$ are both $i \mapsto (0,i)$, which coincides with $\mathcal{S}_{j\cap j'}$. For $[n+1-\delta] \ni i \geq j$, we have $\Theta(i) = i+\delta \geq j' > j$ and so $\mathcal{S}_j\Theta(i) = (1,\Theta(i)-1) = \mathcal{S}_{j'}\Theta(i)$. This coincides with $\mathcal{S}_{j\cap j'}(i) = (1, i-1+\delta)$,[15] which proves the commutativity of the diagram.

We must show, therefore, that the two maps

$$\Theta^*\boxed{\gamma}_{\mathcal{S}_j}, \Theta^*\boxed{\gamma}_{\mathcal{S}_{j'}} : \Delta[n+1-\delta] \to \mathcal{EX}$$

coincide. This will reduce to a question about the behaviour of $\flat(-,-) := \flat_{-,-}$. We observe

$$\begin{aligned}\Theta &= \partial_{j'-1}\partial_{j'-2}\cdots\partial_{j+1}\partial_j \\ &: \Delta[n+1-\delta] \hookrightarrow \Delta[n+1-\delta+1] \hookrightarrow \cdots \hookrightarrow \Delta[n+1]\end{aligned}$$

and note that $\Theta^* = d_j d_{j+1}\cdots d_{j'-2}d_{j'-1} = d_j d_j \cdots d_j d_j$ by repeated application of the simplicial identitiy $d_\alpha d_\beta = d_{\beta-1}d_\alpha$ for $\alpha < \beta$. This implies

$$\begin{aligned}\Theta^*\boxed{\gamma}_{\mathcal{S}_j} &= (d_j d_j \cdots d_j \mathrm{id}\iota\gamma, \flat(\flat(\cdots\flat(j,j'-1),j+1),j)) \\ &= (d_j d_{j+1}\cdots d_{j'-3}d_{j'-2}\iota\gamma, j)\end{aligned}$$

using the simplicial identitiy $d_j s_{j-1} = \mathrm{id}$ and then by repeated un-application of the one mentioned previously. For the exit index, we used that $\flat(\alpha,\beta) = \alpha$ if $\beta \geq \alpha$, so that

$$\flat(\flat(\cdots\flat(j,j'-1),j+1),j) = \flat(\flat(\dots\flat(j,j'-2),j+1),j) = \cdots = \flat(j,j) = j.$$

On the other hand, using $\flat(\alpha,\beta) = \alpha - 1$ if $\beta < \alpha$, we have

$$\flat(\flat(\cdots\flat(j',j'-1),j+1),j) = \flat(\flat(\cdots\flat(j'-1,j'-2),j+1),j) = \cdots = \flat(j+1,j) = j.$$

Now, the simplicial identity $d_\alpha s_\beta = \mathrm{id}$ if $\alpha \in \{\beta, \beta+1\}$ implies $d_{j'-1}s_{j'-1} = \mathrm{id}$, so

$$\begin{aligned}\Theta^*\boxed{\gamma}_{\mathcal{S}_{j'}} &= (d_j \cdots d_{j'-1}s_{j'-1}\iota\gamma, \flat(\flat(\cdots\flat(j',j'-1),j+1),j)) \\ &= (d_j\cdots d_{j'-2}\iota\gamma, j) \\ &= \Theta^*\boxed{\gamma}_{\mathcal{S}_j},\end{aligned}$$

as desired.

As for the second statement, we note simply that the triple

$$(\pi, \Box, \iota)\colon L \to M \times \mathcal{EX}^{\Delta[1]} \times N$$

factorises through $(M \downarrow N)$ by construction, due to (6.6.8) and (6.6.9). $\Box$

[15] We very much do need the commutativity of addition here!

We obtain the *only-if* statement in Theorem 6.4.20 as a consequence.

Corollary 6.6.10. *A cartesian Y-structure $t\colon \mathcal{EX} \to \overline{Y}_{(n,m)}$ on the linked manifold $\mathfrak{S}$ induces a compatibility $\rho_t\colon L \to \mathrm{Hom}(\pi^*t|_M(-), \iota^*t|_N(-))$.*

PROOF. The map t induces a map $(M \downarrow N) \to (t|_M \downarrow t|_N)$. Precomposing with Proposition 6.6.7, we obtain

$$\rho_t = t \circ \Box\colon L \to (t|_M \downarrow t|_N).$$

This factors through $(\pi^*t|_M \downarrow \iota^*t|_N)$ by the very construction of $\Box$: (6.6.9) and (6.6.8) imply $\mathrm{ev}_0\boxed{-} = \pi(-)$ and $\mathrm{ev}_1\boxed{-} = \iota(-)$. $\Box$

6.7. The *if* statement in Theorem 6.4.20

We start with an observation on the relationship between the box construction and (the proof of) Theorem 3.4.1.

Observation 6.7.1. *For $\mathfrak{S}$ a linked space, the box construction $\Box$ itself gives an equivalence $L \to (M \downarrow N)$.*

PROOF. Let $b \in L_n$, and let $\beta\colon \Delta[1]\times\Delta[n] \to N$ be the degenerate composition $\Delta[1] \times \Delta[n] \overset{\mathrm{pr}}{\twoheadrightarrow} \Delta[n] \xrightarrow{\iota(b)} N$, so that $\beta \in (L \downarrow N)$. For $j \in \{1, \dots, n+1\}$, we have $\boxed{b}_{\mathcal{S}_j} = (s_{j-1}\iota(b), j)$, and, for $\Psi\colon (L \downarrow N) \to (M \downarrow N)$ the map from the proof of Lemma 3.4.3,

$$\Psi(\beta)|_{\mathcal{S}_j} = (\beta|_{\mathcal{S}_j}, j) = \left(\left(\Delta[n+1] \overset{\mathcal{S}_j}{\hookrightarrow} \Delta[1] \times \Delta[n] \twoheadrightarrow \Delta[n] \xrightarrow{\iota(b)} N\right), j\right).$$

The underlying simplex map $\mathrm{pr} \circ \mathcal{S}_j\colon [n+1] \to [n]$ is $i \mapsto (0, i) \mapsto i$ if $i \leq j-1$ and $i \mapsto (1, i-1) \mapsto (i-1)$ if $i \geq j$, so $\mathrm{pr} \circ \mathcal{S}_j = s_{j-1}$. Thus

$$\Box = \Psi \circ \mathrm{pr}^*$$

where pr^* is the map $L \mapsto (L \downarrow N)$ that sends b to β, our cylindrical constant-loop inclusion. Now, since the box construction factors as $\Box\colon L \to (M \downarrow N)$, we have the composition $\Phi \circ \Box\colon L \to (L \downarrow N)$, but since this restriction $\Phi\colon (M \downarrow L) \to (L \downarrow N)$ clearly factors through $(L \downarrow L)$, we have the composition $\Phi \circ \Box\colon L \to (L \downarrow L)$. Composing with Ψ finally yields

$$L \overset{\Box}{\to} (M \downarrow L) \overset{\Phi}{\to} (L \downarrow L) \overset{\Psi}{\to} (M \downarrow L).$$

which reproduces in fact the equivalence from Theorem 3.4.1. We have, then, that $\Box$ itself gives an equivalence $L \to (M \downarrow N)$. $\Box$

Definition 6.7.2. Let $n \geq 1$, and let

$$\mathcal{P}^{\Delta}_{n-1,e} \subset \mathcal{P}^{\Delta}_{n-1} \subset \mathcal{EX}_n$$

denote the set of exit n-paths of index $e \in \{1, \dots, n\}$. Further, write

$$\Delta[n]_e \coloneqq (\Delta[1] \times \Delta[e-1]) \amalg_{\Delta[e-1]} \Delta[n]$$

where the gluing is along

$$\Delta\{1\} \times \mathrm{id}\colon \Delta[e-1] \hookrightarrow \Delta[1] \times \Delta[e-1]$$

and
$$\mathrm{id}\colon \Delta[e-1] \hookrightarrow \Delta[n]$$
(cf. (6.6.6) for $n = e = 2$ and (6.6.4) for $n = 2$, $e = 1$).

Lemma 6.7.3. *In the situation of Definition 6.7.2, there is a map*
$$\boxed{\mathbf{b}-}\colon \mathcal{P}^{\Delta}_{n-1,e} \to (M \downarrow N)_{e-1}$$
which for each $(\Gamma, e) \in \mathcal{P}^{\Delta}_{n-1,e}$ *induces a map*
$$\boxed{\mathbf{b}\Gamma_e} \cup \Gamma\colon \Delta[n]_e \to \mathcal{E}\mathcal{X}.$$

Proof. Let
$$\mathbf{b}\Gamma_e \in L_{e-1}$$
be the lift to L of $\Gamma|_{0,\dots,e-1}$ – the latter lies within $\iota(L)$ by the construction of $\mathcal{P}^{\Delta}$ – and set
$$\boxed{\mathbf{b}(\Gamma, e)} := \boxed{\mathbf{b}\Gamma_e} \in (M \downarrow N)_{e-1}$$
using Proposition 6.6.7. Keeping in mind Footnote 12, we now observe that
$$\begin{tikzcd} \Delta[e-1] \arrow[r, hook, "\mathrm{id}"] \arrow[d, hook, "\Delta\{1\}\times\mathrm{id}"'] & \Delta[n] \arrow[d, "\Gamma"] \\ \Delta[1]\times\Delta[e-1] \arrow[r, "\boxed{\mathbf{b}(\Gamma,e)}"'] & \mathcal{E}\mathcal{X} \end{tikzcd}$$
commutes by construction: $\boxed{\mathbf{b}\Gamma_e}_{\Delta\{1\}\times\Delta[n]} = \iota(\mathbf{b}\Gamma_e) = \Gamma|_{0,\dots,e-1}$. This defines $\boxed{\mathbf{b}\Gamma_e} \cup \Gamma\colon \Delta[n]_e \to \mathcal{E}\mathcal{X}$. $\square$

Remark 6.7.4. Note that we have not used in the proof of Proposition 6.6.7 that $\mathfrak{S}$ is a linked manifold. Thus, the box construction gives a map $\mathcal{L} \to \mathcal{E}\mathcal{X}^{\Delta[1]}$ for any linked ∞-category $\mathfrak{S} = (\mathcal{M} \leftarrow \mathcal{L} \rightarrow \mathcal{N})$. Lemma 6.7.3 also remains true in this generality.

Notation 6.7.5. For $(\Gamma, e) \in \mathcal{P}^{\Delta}_{n-1}$, we write $\boxed{(\Gamma, e)} := \boxed{\Gamma, e} := \boxed{\mathbf{b}\Gamma_e} \cup \Gamma$. This defines a function
$$\square\colon \mathcal{P}^{\Delta}_{n-1} \to \coprod_{e\in\{1,\dots,n\}} \mathrm{Hom}(\Delta[n]_e, \mathcal{E}\mathcal{X}),$$
which in turn yields a map
$$\square\colon \mathcal{E}\mathcal{X}_n = M_n \amalg \mathcal{P}^{\Delta}_{n-1} \amalg N_n \to M_n \amalg \left(\coprod_{e\in\{1,\dots,n\}} \mathrm{Hom}(\Delta[n]_e, \mathcal{E}\mathcal{X}) \right) \amalg N_n$$
by declaring it to be the identity on $M, N \subset \mathcal{E}\mathcal{X}$. (Here, $\mathcal{P}^{\Delta}_{-1} = \emptyset$ is understood.) We write
$$\mathcal{P}^{\square}_{n-1,e} := \boxed{\mathcal{P}^{\Delta}_{n-1,e}}, \ \mathcal{P}^{\square}_{n-1} := \boxed{\mathcal{P}^{\Delta}_{n-1}}$$
for the images.

Definition 6.7.6. The *boxed exit path ∞-category* $\mathcal{EX}^{\square}(\mathfrak{S})$ of a linked ∞-category $\mathfrak{S}$ is defined, using Notation 6.7.5, by

$$\begin{aligned}\mathcal{EX}^{\square}_n &:= \boxed{\mathcal{EX}_n} = \mathcal{M}_n \amalg \mathcal{P}^{\square}_{n-1} \amalg \mathcal{N}_n \\ &\subset \mathcal{M}_n \amalg \left(\coprod_{e \in \{1,\dots,n\}} \operatorname{Hom}(\Delta[n]_e, \mathcal{EX}) \right) \amalg \mathcal{N}_n.\end{aligned}$$

We declare the face and degeneracy maps to be those inherited from $\mathcal{EX}$:

$$d_i \boxed{\alpha} = \boxed{d_i\alpha}, \;\; s_i \boxed{\alpha} = \boxed{s_i\alpha}.$$

Proof that $\mathcal{EX}^{\square}$ is an ∞-category. It suffices to note that $\square$ of Notation 6.7.5 is injective for any $n \geq 0$: if $\Gamma \neq \Gamma'$, then the restrictions along $\Delta[n] \hookrightarrow \Delta[n]_e$ of $\boxed{\Gamma, e}$, $\boxed{\Gamma', e'}$ differ; and if $e \neq e'$, then they are already in different connected components. This justifies the above definition in the first place by making d_i and s_i well-defined. $\square$

Lemma 6.7.7. *For any linked ∞-category, we have $\mathcal{EX} \cong \mathcal{EX}^{\square}$. Consequently, any map $f\colon \mathfrak{S} \to \mathfrak{T}$ of linked ∞-categories induces an ∞-functor $f\colon \mathcal{EX}^{\square}(\mathfrak{S}) \to \mathcal{EX}^{\square}(\mathfrak{T})$.*

Proof. The isomorphism $\to$ is given by the map $\square$ itself, which is functorial by the construction of $\mathcal{EX}^{\square}$. The first statement follows from its dimension-wise injectivity as noted above, and the second statement is trivial. $\square$

Let now $F\colon Y \to BO(n+m)$ be a smooth tangential structure, and let $\mathfrak{S}$ be a linked manifold of top dimension $n+m$, together with a solid Y-structure consisting of $t_M\colon M \to \overline{Y}|_n$, $t_N\colon N \to \overline{Y}|_{n+m}$, and $\rho\colon L \to (\pi^* t_M \downarrow \iota^* t_N)$.

Assume moreover that ρ is induced by a compatibility of the form $\rho\colon L \to BO(n+m)^{\Delta^2}|^Y_{\oplus}$ as in Lemma 6.4.15, which is to say that at $\ell \in L$ the edge

$$(\pi^*(W \oplus \mathrm{T}M) \to \iota^*(\mathrm{T}N))_\ell = (W_{\pi(\ell)} \oplus \mathrm{T}_{\pi(\ell)}M \to \mathrm{T}_{\iota(\ell)}N)$$

(cf. (6.4.13)) is constant in $BO(n+m)$.

Finally, let us write $\mathrm{Ar}\mathcal{C} := \mathrm{Ar}(\mathcal{C}) := \mathcal{C}^{\Delta[1]}$.

Lemma 6.7.8. *Let $\mathfrak{S}$ and its solid Y-structure be as above. Then each $(\Gamma, e) \in \mathcal{P}^{\Delta}_{n-1,e}$ induces a map*

$$t^e(\Gamma, e)\colon \Delta[n]_e \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}.$$

such that

(1) $t^e(\Gamma, e)|_{\Delta\{0\} \times \Delta[e-1]} = t_M(\pi(\mathbf{b}\Gamma_e))$,
(2) $t^e(\Gamma, e)|_{\Delta[n]} = t_N(\Gamma)$.

Proof. Let us first consider the lowest dimension: let $n = 1$ and so $e = 1$. Let $(\Gamma, 1) \in \mathcal{P}^{\Delta}_{0,1} = \mathcal{P}^{\Delta}_0$ and so $\mathbf{b}\Gamma_1 \in L_0$. We have

$$\Delta[1]_1 \cong \Delta[1] \vee \Delta[1]$$

and $\boxed{\mathbf{b}\Gamma_e} \cup \Gamma$ is given by

$$(s_0 \iota(\mathbf{b}\Gamma_1), 1) \vee \Gamma\colon \Delta[1] \vee \Delta[1] \to \mathcal{EX}.$$

The compatibility is given as a map $\rho\colon L \to (\pi^* t_M \downarrow \iota^* t_N) \hookrightarrow (\mathrm{Ar}\mathcal{V}^{\hookrightarrow})^{\Delta[1]}$. Let us, then, set

$$t^e(\Gamma, e) = \rho(\mathbf{b}\Gamma_1) \vee t_N(\Gamma)\colon \Delta[1] \vee \Delta[1] \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}.$$

This is well-defined. Indeed, the first component is of the form

$$(\rho(\mathbf{b}\Gamma_1)\colon \Delta[1] \times \Delta\{0\} \simeq \Delta[1] \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}) \in \big((\mathrm{Ar}\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\big)_0$$

with target

$$(\mathrm{ev}_1\rho(\mathbf{b}\Gamma_1) = \iota^* t_N(\mathbf{b}\Gamma_1)\colon \mathrm{T}_{\iota(\mathbf{b}\Gamma_1)}N \to F^N_{\iota(\mathbf{b}\Gamma_1)}) \in (\mathrm{Ar}\mathcal{V}^{\hookrightarrow})_0 = \mathrm{Hom}(\Delta[1], \mathcal{V}^{\hookrightarrow})$$

by construction (Item 2 of Lemma 6.4.12). This is precisely the value of the second component $t_N(\Gamma)$ at the gluing vertex:

$$d_1 t_N(\Gamma) = t_N(d_1\Gamma) = t_N(\iota(\mathbf{b}\Gamma_1))$$

since $\iota(\mathbf{b}\Gamma_1) = \Gamma|_0$ by construction (see the proof of Lemma 6.7.3). That we have fulfilled Condition 2 is clear. For Condition 1, note similarly that

$$\mathrm{ev}_0\rho(\mathbf{b}\Gamma_1) = \pi^* t_M(\mathbf{b}\Gamma_1) = t_M(\pi(\mathbf{b}\Gamma_1))$$

by Item 1 of Lemma 6.4.12.

The general case is entirely analogous: we set

$$t^e(\Gamma, e) = \rho(\mathbf{b}\Gamma_e) \cup t_N(\Gamma)\colon \Delta[n]_e \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}.$$

This is well-defined: since $\mathbf{b}\Gamma_e \in L_{e-1}$, ρ thereof is of type $\Delta[1] \times \Delta[e-1] \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}$. Further, its target $(e-1)$-simplex is

$$\mathrm{ev}_1\rho(\mathbf{b}\Gamma_e) = \iota^* t_N(\mathbf{b}\Gamma_e) = t_N(\Gamma|_{0,\dots,e-1}) = t_N(\Gamma)|_{0,\dots,e-1},$$

so that the two components glue. Finally, that the two conditions are fulfilled can be seen in the same way as above. □

Now, Lemma 6.7.8 can be restated as saying that a solid Y-structure on a linked manifold $\mathfrak{S}$ induces a map

$$t^e\colon \mathcal{P}^{\square}_{n-1,e} \to \mathrm{Hom}(\Delta[n]_e, \mathrm{Ar}\mathcal{V}^{\hookrightarrow}).$$

This systematises homotopy-commuting diagrams like (6.5.7).

Lemma 6.7.9. *The map t^e factors as*

$$\mathcal{P}^{\square}_{n-1,e} \to \mathrm{Hom}(\Delta[n]_e, \overline{Y}_{(n,m)}) \to \mathrm{Hom}(\Delta[n]_e, \mathrm{Ar}\mathcal{V}^{\hookrightarrow})$$

along $\overline{Y}_{(n,m)} \hookrightarrow \mathrm{Ar}\mathcal{V}^{\hookrightarrow}$.

Proof. The restriction of t^e along

$$(\Delta[1] \times \Delta[e-1] \hookrightarrow \Delta[n]_e \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}) \in (\mathrm{Ar}\mathcal{V}^{\hookrightarrow})^{\Delta[1]}_{e-1}$$

lifts to

$$(\overline{Y}_{(n,m)})^{\Delta[1]} \hookrightarrow (\mathrm{Ar}\mathcal{V}^{\hookrightarrow})^{\Delta[1]}$$

by Lemma 6.4.12, and so does the restriction along

$$\Delta[n] \hookrightarrow \Delta[n]_e \to \mathrm{Ar}\mathcal{V}^{\hookrightarrow}$$

by the definition of t_N. □

This suggests that t_M, t_N, and the t^e should couple together to give a map from $\mathcal{EX}^\square$ (or equivalently $\mathcal{EX}$) to a box-like variant of $\overline{Y}_{(n,m)}$. We will now set up such a version.

In order to do so, we will relax the trivial idea of Notation 6.7.5 and Definition 6.7.6 to a slightly less trivial one. In the former, non-invertible n-simplices are not of type $\Delta[n]_e \to \mathcal{EX}$ generally, but are only those that are given by the box construction. Where the simplex $\{1\} \times \Delta[e-1]$ is mapped to is determined exactly by the link embedding $\iota\colon L \hookrightarrow N$. In an ∞-category that is not given as an exit path ∞-category $\mathcal{C}$ of a span, however, there is no such a priori information. However, if $\mathcal{C}$ is 'exit-type,' we can imitate this construction without having to reduce fully to a span. We leave this to future work and will content ourselves with treating the categories at hand.

Definition 6.7.10. Let $\mathcal{V}^{\hookrightarrow}|_{n,m}$ be as in Corollary 6.3.12, and consider the simplicial set $\boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}$ whose objects are those of $\mathcal{V}^{\hookrightarrow}|_{n,m}$, and, for $k \geq 1$,

$$\boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}_k := B\mathrm{O}(n) \amalg \left(\coprod_{e\in\{1,\ldots,k\}} \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k]_e, \mathcal{V}^{\hookrightarrow}) \right) \amalg B\mathrm{O}(n+m)$$

where

$$\begin{gathered}\mathrm{Hom}^{n,m}_{\mathrm{sSet}}\left(\Delta[k]_e, \mathcal{V}^{\hookrightarrow}\right) := \\ \left(B\mathrm{O}(n) \downarrow B\mathrm{O}(m) \oplus B\mathrm{O}(n)\right)^{\mathcal{V}^{\hookrightarrow}}_{e-1} \times_{\mathrm{Ar}(\mathcal{V}^{\hookrightarrow})_{e-1}} \mathrm{Hom}_{\mathrm{sSet}}\left(\Delta[n]_e, \mathcal{V}^{\hookrightarrow}\right) \times_{\mathcal{V}^{\hookrightarrow}_n} B\mathrm{O}(n+m)_k \\ = \\ \big\{\alpha\colon \Delta[k]_e \to \mathcal{V}^{\hookrightarrow} : \alpha|_{\Delta[k]} \in B\mathrm{O}(n+m)_k, \\ \alpha|_{\Delta[1]\times\Delta[e-1]} \in \left(B\mathrm{O}(n) \downarrow B\mathrm{O}(m) \oplus B\mathrm{O}(n)\right)^{\mathcal{V}^{\hookrightarrow}}_{e-1}\big\}.\end{gathered}$$

There are evident maps

$$d_i\colon \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k]_e, \mathcal{V}^{\hookrightarrow}) \to \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k-1]_{\flat_{e,i}}, \mathcal{V}^{\hookrightarrow})$$

when the i'th face is vertical in the obvious sense, and maps

$$d_i\colon \mathrm{Hom}^{n,m}_{\mathrm{sSet}}\left(\Delta[k]_e, \mathcal{V}^{\hookrightarrow}\right) \to \boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}^{\simeq}_{k-1} = B\mathrm{O}(n)_{k-1} \amalg B\mathrm{O}(n+m)_{k-1}$$

if it is low or upper instead; and maps

$$s_i\colon \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k]_e, \mathcal{V}^{\hookrightarrow}) \to \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k+1]_{\sharp_{e,i}}, \mathcal{V}^{\hookrightarrow})$$

which all together make $\boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}$ an ∞-category, as can be proved analogously to Theorem 3.2.11.

There is a generalisation of the idea of Remark 5.2.53 that gives a map of type $\mathcal{V}^{\hookrightarrow}|_{n,m} \to \boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}$:

Construction 6.7.11. Let $\Gamma\colon \mathrm{Path}[k+1] \to B^{\oplus}\mathrm{O}$ be a k-simplex of $\mathcal{V}^{\hookrightarrow}|_{n,m}$ whose vertices hit both $B\mathrm{O}(n)$ and $B\mathrm{O}(n+m)$. Then there is a unique natural number $e \in \{1, \ldots, k\}$ such that $\Gamma(\nabla)|_{0,\ldots,e-1} \in B\mathrm{O}(m) \oplus B\mathrm{O}(n)$ and

$\Gamma(\nabla)|_{e,\ldots,k} \in BO(n+m)$,[16] where $\Gamma(\nabla) \in BO(n+m)$ is as in Remark 5.2.53 using Construction 5.2.31.

We may now imitate Lemma 6.7.3 and Notation 6.7.5: using the coordinate projection $BO(m) \oplus BO(n) \to BO(n)$, we obtain

$$\boxed{\Gamma(\nabla)|_{0,\ldots,e-1}} \in \left((BO(n) \downarrow BO(n+m))^{\mathcal{V}^{\hookrightarrow}|_{n,m}}\right)_{e-1},$$

which glues along its restriction to its top side $\Delta\{1\} \times \Delta[e-1]$ with $\Gamma(\nabla)$ itself, so that we have constructed

$$\boxed{\Gamma(\nabla)} := \boxed{\Gamma(\nabla)|_{0,\ldots,e-1}} \cup \Gamma(\nabla).$$

Proposition 6.7.12. *The rule* $\Gamma \mapsto \boxed{\Gamma(\nabla)}$ *of Construction 6.7.11 lifts to an equivalence*

$$\boxed{\nabla^*}\colon \mathcal{V}^{\hookrightarrow}|_{n,m} \xrightarrow{\sim} \boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}.$$

Proof. If $\Gamma \in \mathcal{V}^{\hookrightarrow}_k$ lies entirely within $BO(n)$ or $BO(n+m)$, then we let $\boxed{\nabla^*} = \nabla^*$ be the map of Remark 5.2.53. The rule is evidently simplicial and a bijection on objects, so we need only show that it induces equivalences on morphism spaces. Temporarily, let us write $\mathbf{V} := \mathcal{V}^{\hookrightarrow}|_{n,m}$ and $\boxed{\mathbf{V}} := \boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}}$, and take $V \in BO(n)$, $K \in BO(n+m)$.

Similarly to the proof of Theorem 3.3.1, we pass to the right-pinched model and obtain $\mathrm{Hom}_{\boxed{\mathbf{V}}}(V,K) \simeq \mathrm{Hom}^{\mathrm{R}}_{\boxed{\mathbf{V}}}(V,K) = \{V\} \times_{\boxed{\mathbf{V}}} \boxed{\mathbf{V}}/K$. In order to write this in terms of $\mathbf{V}$ instead, note first that the right-pinched model restricts us again to maximal exit index, so at each level k, we need only consider $e = k+1$; that is, there are natural bijections

$$\left(\{V\} \times_{\boxed{\mathbf{V}}} \boxed{\mathbf{V}}/K\right)_k \cong \{V\} \times_{BO(n)_k} \mathrm{Hom}^{n,m}_{\mathrm{sSet}}(\Delta[k+1]_{k+1}, \mathbf{V})$$

where the morphism set projects to $BO(n)$ by restricting along the 'bottom-side' inclusion $\Delta[k] \cong \Delta\{0\} \times \Delta[k] \hookrightarrow \Delta[k+1]_{k+1}$.

We can now turn the (ordinary) colimit defining $\Delta[k]_e$ into a(n ordinary) limit of morphism spaces, so that the resulting natural bijections yield an isomorphism

$$\mathrm{Hom}^{\mathrm{R}}_{\boxed{\mathbf{V}}}(V,K) \cong \{V\} \times_{BO(n)} (BO(n) \downarrow BO(m) \oplus BO(n)) \times_{\mathbf{V}} \mathbf{V}/K$$

of simplicial sets, where the fibre product over $\mathbf{V}$ is taken on the right with respect to the canonical right-fibration $\mathbf{V}/K \to \mathbf{V}$ and on the left with respect to the 'top-side' evaluation $(BO(n) \downarrow BO(m) \oplus BO(n)) \xrightarrow{\mathrm{ev}_1} \mathbf{V}$; and the fibre product over $BO(n)$ is taken with respect to the 'bottom-side' evaluation.

[16]We can allow Γ to lie wholly within $BO(n)$ or $BO(n+m)_k$ in an evident manner by considering $e \in \{0,\ldots,k+1\}$.

We conclude by observing that $(B\mathrm{O}(n) \downarrow B\mathrm{O}(m) \oplus B\mathrm{O}(n)) \simeq B\mathrm{O}(m) \times B\mathrm{O}(n)$ by Theorem 3.4.1, and so

$$\begin{aligned}
\mathrm{Hom}^{\mathrm{R}}_{\boxed{\mathbf{V}}}(V,K) &\simeq \{V\} \times_{B\mathrm{O}(n)} (B\mathrm{O}(m) \times B\mathrm{O}(n)) \times_{\mathbf{V}} \mathbf{V}/K \\
&\cong (B\mathrm{O}(m) \oplus \{V\}) \times_{\mathbf{V}} \mathbf{V}/K \\
&\simeq P(B\mathrm{O}(n+m))_{B\mathrm{O}(m)\oplus\{V\},K} \\
&\simeq \mathrm{Hom}_{\mathbf{V}}(V,K)
\end{aligned}$$

using the calculation in the proof of Lemma 6.3.10, and where the map $B\mathrm{O}(m) \times B\mathrm{O}(n) \to \mathbf{V}$ is $\oplus$, factoring through $B\mathrm{O}(n+m)$. The map $\boxed{\nabla^*}$ induces precisely the inverse equivalence $(B\mathrm{O}(m) \oplus \{V\}) \times_{\mathbf{V}} \mathbf{V}/K \to \mathrm{Hom}^{\mathrm{R}}_{\boxed{\mathbf{V}}}(V,K)$. $\square$

Corollary 6.7.13. *A solid Y-structure on a linked manifold $\mathfrak{S}$ induces a cartesian Y-structure on $\mathfrak{S}$.*

Proof. Using Proposition 6.7.12, we may pull back $\overline{Y}_{(n,m)} \hookrightarrow \mathrm{Ar}\mathcal{V}^{\hookrightarrow}$, which is already over $\mathcal{V}^{\hookrightarrow}|_{n,m}$, along $\boxed{\mathcal{V}^{\hookrightarrow}|_{n,m}} \simeq \mathcal{V}^{\hookrightarrow}|_{n,m}$, and write $\boxed{Y_{(n,m)}}$ for the result. Now, Lemma 6.7.9 can be restated as saying that the solid structure on $\mathfrak{S}$ induces a lift $\mathcal{E}\mathcal{X}^{\square}(\mathfrak{S}) \to \boxed{Y_{(n,m)}}$, which concludes the proof by Lemma 6.7.7. $\square$

Remark 6.7.14. The proof of Theorem 6.4.20 does not depend on our assumption that the rank of the smooth tangential structure coincide with the bulk dimension of the linked space in question: this only simplified some notation. The statement is therefore true in its stated generality. The corresponding modifications are clear – for instance, in Definition 6.4.18, the structure on N will be determined by a bundle isomorphism $\mathrm{T}N \oplus Z \cong F^N$, and we will ask for an isomorphism $\pi^*W \cong \mathrm{N}_N M \oplus \iota^* Z$, and again continue with an equality of classifiers using Lemma 6.1.1.

CHAPTER 7

Divide and conquer

7.1. Maps of linked spaces

In this section, we propose a notion of a map of linked spaces. Span maps clearly induce maps between the respective exit path ∞-categories. By leveraging Theorem 3.3.1, we will suggest a simple definition of maps of linked spaces in general. Any such notion should be well-behaved in two respects:

- maps should compose, and
- they should induce functors on exit paths ∞-categories.

Our proposal fulfills both criteria (by Remark 7.1.4 and Proposition 7.1.10, respectively) and is fairly straightforward. We will start with an obvious extension of Definition 3.2.14 that lets us deal with more strata while retaining depth 1, and our exit path ∞-category construction of Definition 3.2.2 extends in the obvious manner.

The only new part of the definition is that we also assign spans over individual elements of the stratifying poset, with the consequence that we may now also consider maps from non-trivially linked spaces to ordinary (non-stratified) spaces. The definition we give is slightly more relaxed than that of ordinary stratified maps, but it does reproduce stratified maps: see Remark 7.1.12.

Definition 7.1.1. A *linked space*, denoted by

$$M \to \mathfrak{P},$$

of *depth* 1 is a collection of spaces indexed over a poset $\mathfrak{P}$ of depth 1: we have a space $M_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathfrak{P}$, and for each arrow $\mathfrak{p} \leq \mathfrak{q}$ a space $L_{\mathfrak{pq}}$, called a *link*, sitting in a span

$$M_p \xleftarrow{\pi} L_{pq} \xrightarrow{\iota} M_q,$$

such that

- if $\mathfrak{p} \leq \mathfrak{q}$ but $\mathfrak{p} \neq \mathfrak{q}$, then π is a proper fibre bundle and ι a (closed) embedding;[1]
- if $\mathfrak{p} = \mathfrak{q}$, then
$$L_{\mathfrak{pp}} = M_{\mathfrak{p}}^I,$$
the unbased path space of $M_{\mathfrak{p}}$ ($I = [0, 1]$). The exponential is endowed with the compact-open topology, and
$$\pi = \mathrm{ev}_0, \quad \iota = \mathrm{ev}_1$$

[1]The closedness requirement is somewhat superfluous in general: the construction of $\mathcal{EX}$ requires only that $\mathrm{Sing}_\bullet(\iota)$ be a monomorphism. Still, going backwards may be problematic; cf. the proof of Lemma 6.1.1.

are, respectively, the source and target evaluations.

We call $M \to \mathfrak{P}$ a *linked manifold* (of *depth* 1) if all $M_\mathfrak{p}$, as well as $L_{\mathfrak{pq}}$ whenever $\mathfrak{p} \neq \mathfrak{q}$, are smooth riemannian manifolds.

The appropriate construction of $\mathcal{EX}(M \to \mathfrak{P})$ is entirely analogous to the one we've given, which is the case $|\mathfrak{P}| = 2$. The case of higher depth, which is naturally more complicated, will appear elsewhere – see Section 7.3 for an idea (though not for the exit path construction), and Section 3.5 for a suggestion to circumvent a native higher-depth treatment by *iterating* the depth-one exit path construction.

The definition of a *linked* ∞*-category* of *depth* 1 is clear from the definition above: the π are required to be right fibrations, the ι cofibrations, and the $L_{\mathfrak{pp}} = M_\mathfrak{p}^I$ are replaced by the arrow ∞-category $\mathrm{Ar}\mathcal{M}_\mathfrak{p} = \mathcal{M}_\mathfrak{p}^{\Delta[1]}$. The following definition also applies in this generality.

Definition 7.1.2. A *map* f of linked spaces, written

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ \mathfrak{P} \arrow[r] & \mathfrak{Q} \end{tikzcd}, \tag{7.1.3}$$

consists of a map of posets $\mathfrak{f}\colon \mathfrak{P} \to \mathfrak{Q}$ together with maps of spans

$$f_{\mathfrak{pq}}\colon (M_\mathfrak{p} \leftarrow L_{\mathfrak{pq}} \to M_\mathfrak{q}) \to (N_{\mathfrak{f}(\mathfrak{q})} \leftarrow L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \to N_{\mathfrak{f}(\mathfrak{q})})$$

for each pair $\mathfrak{p} \leq \mathfrak{q}$ in $\mathfrak{P}$. This means that $f_{\mathfrak{pq}}$, with a slight abuse of notation, is a commutative diagram of the form

$$\begin{tikzcd} L_{\mathfrak{pq}} \arrow[r, "f_{\mathfrak{pq}}"] \arrow[d, "\pi\times\iota"'] & L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \arrow[d, "\pi'\times\iota'"] \\ M_\mathfrak{p} \times M_\mathfrak{q} \arrow[r, "f_\mathfrak{p}\times f_\mathfrak{q}"] & N_{\mathfrak{f}(\mathfrak{p})} \times N_{\mathfrak{f}(\mathfrak{q})} \end{tikzcd}$$

with no further compatibility conditions since we are in depth 1 (but cf. (7.3.1) for depth 2).

Remark 7.1.4. Since both poset maps and span maps compose, maps of linked spaces compose.

Warning 7.1.5. Diagrams of type (7.1.3), while literal in stratified geometry, are only figurative in the linked context!

We have thus obtained a notion of mapping from a non-trivially linked space, say of type $(M_0 \leftarrow L = L_{01} \to M_1) = (M \to \{0 < 1\})$ to a smooth space X. The latter is naturally a linked space indexed over the trivial poset $*$, so that the its full expression is $X \xleftarrow{\mathrm{ev}_0} X^I \xrightarrow{\mathrm{ev}_1} X$. Thus, a map

$$\begin{tikzcd} M \arrow[r] \arrow[d] & X \arrow[d] \\ \{0<1\} \arrow[r] & * \end{tikzcd} \tag{7.1.6}$$

is a commuting square

$$\begin{tikzcd} L \arrow[r, "f_L := f_{01}"] \arrow[d, "\pi \times \iota"'] & X^I \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\ M_0 \times M_1 \arrow[r, "f_0 \times f_1"] & X \times X \end{tikzcd} . \tag{7.1.7}$$

The sector of (7.1.6) over the two identities in $\{0 < 1\}$,

$$\begin{tikzcd} M_0^I \arrow[r] \arrow[d] & X^I \arrow[d] \\ M_0 \times M_0 \arrow[r] & X \times X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_1^I \arrow[r] \arrow[d] & X^I \arrow[d] \\ M_1 \times M_1 \arrow[r] & X \times X \end{tikzcd} ,$$

is determined by f_0 and f_1.

What one might object to is that if the linked spaces in question are induced by depth-1 stratified spaces, then maps in our sense are more relaxed than stratified maps of the corresponding spaces: the latter give only a subset of the former. While this relaxation poses no problem for our purposes, let us illustrate it with the simplest example where it is detectable.

Example 7.1.8 (linked paths in a smooth space)**.** Let $M \to \mathfrak{P}$ be induced by $\mathbf{R}_{\geq 0}$ with its standard stratification, i.e., $M_0 = \{0\}$, $M_1 = \mathbf{R}_{>0}$, $L = L_{01} = *$, with $\iota = \iota_+ \colon L = * \hookrightarrow \mathbf{R}_{>0}$ is given by the choice of some point $+ \in \mathbf{R}_{>0}$. Let also $X \to *$ be induced by a smooth manifold X as above. Then

$$\begin{tikzcd} * \arrow[r, "\gamma := f_{01}"] \arrow[d, "\mathrm{id} \times \iota_+"'] & X^I \arrow[d] \\ \{0\} \times \mathbf{R}_{>0} \arrow[r, "f_0 \times f_1"] & X \times X \end{tikzcd}$$

is determined, besides the stratum-wise maps, by the choice of a single path

$$\gamma \colon f_0(0) \to f_1(+)$$

in X.

Similarly, if $[0,1] \to \{0 < i > 1\}$ (i for *i*nterior) is the linked space of depth 1 with three strata induced by the obvious stratification on $[0,1]$,[2] with both links given again by $*$ and the embeddings into $(0,1)$ determined by a pair of points $\varepsilon < \delta$ in $(0,1)$, then a linked map

$$\begin{tikzcd} {[0,1]} \arrow[r] \arrow[d] & X^I \arrow[d] \\ \{0 < i > 1\} \arrow[r] & * \end{tikzcd} \tag{7.1.9}$$

[2]This is *not* a poset-stratified space in the usual sense, since $[0,1] \to \{0 < i > 1\}$ is not continuous with respect to the Aleksandrov topology on the target. We see in particular that Definition 7.1.1 extends ordinary stratified spaces non-trivially but without losing access to topology.

is determined, besides the stratum-wise maps, by two paths

$$\gamma_\varepsilon \colon f_0(0) \to f_i(\varepsilon),$$
$$\gamma_\delta \colon f_1(1) \to f_i(\delta)$$

in X (see Figure 1).

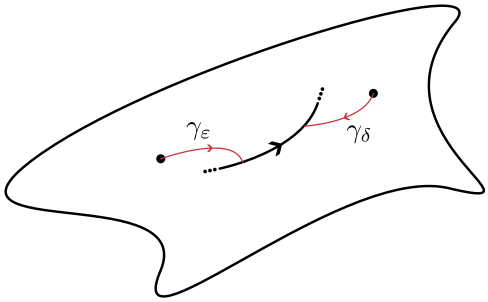


FIGURE 1. A general closed linked path as in (7.1.9).

Proposition 7.1.10. *Given a map f of type*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ \mathfrak{P} \arrow[r] & \mathfrak{Q} \end{tikzcd}$$

in the sense of Definition 7.1.2, there is an induced map

$$f = f_* \colon \mathcal{EX}(M \to \mathfrak{P}) \to \mathcal{EX}(N \to \mathfrak{Q})$$

of exit path ∞-categories. In fact, this defines a(n ordinary) functor

$$\mathrm{LS} \to \mathrm{Cat}_\infty$$

from linked spaces to ∞-categories and ∞-functors.

PROOF. We will describe the construction only at the level of the corresponding topological categories.[3] Without loss of generality, assume $|\mathfrak{P}| = 2$, since $\mathcal{EX}(M \to \mathfrak{P})$ otherwise splits dimension-wise into disjoint unions according to pairs of neighbouring strata of differing depth. The only non-obvious case is when $\mathfrak{Q} = *$, $X := N_*$. Let then $\mathfrak{p} \neq \mathfrak{q}$ in $\mathfrak{P}$, and $p \in M_\mathfrak{p}$, $q \in M_\mathfrak{q}$. We observe f_* in terms of the corresponding locally-Kan categories: on non-invertible paths, we will provide maps

$$\mathrm{Hom}_{\mathcal{EX}(M)}(p,q) \overset{\text{Thm}}{\underset{3.3.1}{\simeq}} \mathcal{P}(M_\mathfrak{q})_{(L_{\mathfrak{p}\mathfrak{q}})_p,q} \to \mathcal{P}(X)_{f(p),f(q)} \simeq \mathrm{Hom}_{\mathrm{Sing}(X)}(f(p),f(q)) \tag{7.1.11}$$

[3]Due to some contractible choices, a full simplicial construction requires equivalent boxed replacements, and the proof technique of Corollary 6.7.13 applies mutatis mutandis. As this will not be needed, we leave it to the interested reader. The proof sketch provided here is an approximation in the spirit of Section 6.5.

while the commuting square (7.1.7) is given. The latter, however, induces, after pulling back onto $p \in \mathfrak{M}_\mathfrak{p}$ (and writing $L \coloneqq L_{\mathfrak{pq}}$), the square

$$\begin{tikzcd} L_p \arrow[r, "f_L"] \arrow[d] & X^I \arrow[d] \\ \{p\} \times M_\mathfrak{q} \arrow[r] & \{f(p)\} \times X \end{tikzcd} .$$

Thus, the initial point γ_0 of a path $\gamma \in \mathcal{P}(M_\mathfrak{q})_{L_p,q}$ yields the path $f_L(\gamma_0)$ in X from $f(p)$ to γ_0, so

$$f_* \colon \gamma \mapsto f_L(\gamma_0) * \gamma$$

provides (7.1.11), as desired (see Figure 2). (We have not distinguished L and

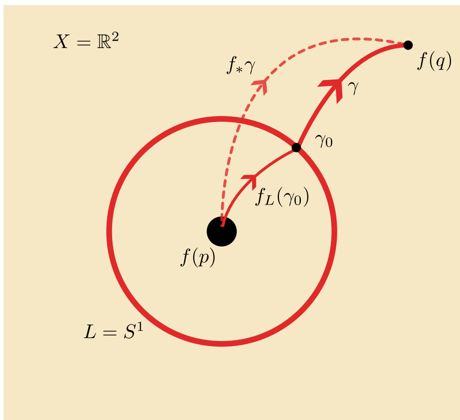


FIGURE 2. The proof of Proposition 7.1.10 for $(M \to \mathfrak{P}) = (\{0\} \twoheadleftarrow S^1 \hookrightarrow \mathbf{R}^2 \smallsetminus \{0\})$ and $N = X = \mathbf{R}^2$, with f 'the identity'; cf. Remark 7.1.12.

$\iota(L) \subseteq M_\mathfrak{q}$ in notation.) The second statement is left to the reader. $\square$

Finally, we note, as proof of concept, that stratified maps induce linked maps.

Remark 7.1.12 (linked maps from stratified maps)**.** Consider a link projection $\pi \colon L \twoheadrightarrow M$. The *fibrewise open cone* on π is defined to be

$$C(\pi) = L \times \mathbf{R}_{\geq 0} \amalg_{L \times \{0\}} M. \tag{7.1.13}$$

Suppose that a linked space $\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N\right)$ is *induced by* a (conically-smooth) stratified space X over $\{0 < 1\}$ with strata $X_0 = M$ and $X_1 = X \smallsetminus X_0 = N$ in the sense that

$$X \cong C(\pi) \amalg_{L \times \mathbf{R}_{>0}} N.$$

and ι by the implicitly used open embedding

$$L \times \mathbf{R}_{>0} \hookrightarrow N$$

at a fixed positive time, say 1. This construction appears in [6, the proof of Lemma 6.1.7]. We interpret it also as a way to naturally associate a (conically-smooth) stratified space to any linked space.

Now, let Y be smooth and $f\colon X \to Y$ a stratified map. There is an induced map of linked spaces $\mathfrak{S} \to Y$ whose non-obvious component $f_L\colon L \to Y^I$ covering $f|_0 \times f|_1\colon M \times N \to Y \times Y$ can be given, at $\gamma_0 \in L$ (using notation from the proof of Proposition 7.1.10), by simply following along the time coordinate in $C(\pi)$ from 0 to 1, from $f|_0(\pi(\gamma_0))$ to $\iota(\gamma_0)$.

The construction we recalled in Remark 7.1.12 is key to relating linked spaces to ordinary stratified spaces. We fix it in a definition for reference.

Definition 7.1.14. Let $\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N\right)$ be a constructible linked manifold (see Definition 6.1.10). Then there is an open embedding $L \times \mathbf{R}_{>0} \hookrightarrow N$, and we call the resulting space

$$|\mathfrak{S}| \coloneqq C(\pi) \amalg_{L \times \mathbf{R}_{>0}} N,$$

with $C(\pi)$ defined by (7.1.13), its *(linked) realisation.* It is naturally stratified over $[1]$ with the cone locus being the 0-stratum and its complement the 1-stratum.

Remark 7.1.15. The linked space associated with the realisation of a constructible linked manifold is the original constructible linked manifold itself.

7.2. Duals of bordisms

The rest of this chapter should be read as a continuation of Section 1.3, so we assume its content.

Notation 7.2.1. In this section, we will resume the notation $\mathbf{n} \coloneqq \mathbf{R}^n$. This will help clarify the different roles of some $\mathbf{R}$-factors that appear below.

The following definition reproduces well-known pictures from [67]. Besides being in a different context – that is, besides working with manifolds with boundaries in the ordinary sense – and at this stage not referring to tangential structure, it is, in essence, not new. The usefulness of this simplistic point of view will start manifesting itself first in our treatment of cutting and gluing in Section 7.5, which in essence also is not new; then in Section 7.4 when it will let us extend these ideas to encompass defect submanifolds, and then also tangential structures.

Definition 7.2.2 (The *Philip II of Macedon* (P^2) construction on bordisms)**.** Let $\mathfrak{M} \to \mathfrak{P}$ be the linked manifold of depth 1 given by a manifold M with boundary $\partial \coloneqq \partial M$ (see Example 3.2.15). We associate with $\mathfrak{M}$ a map

$$p\colon \overline{\mathfrak{M}} \to \mathfrak{M}^!$$

of linked manifolds, defined as follows.

Let $\overline{\mathfrak{P}}$ be the poset with an element 0 and two elements i, i_+ for each connected component ∂_i of $\partial = \coprod_i \partial_i$, together with arrows $i < 0$ and $i < i_+$

and no other non-identity arrows. Further, let $\mathfrak{P}^!$ be the poset containing 0 and every i, *but with reversed arrows* $0 < i$. Evidently, $\overline{\mathfrak{P}}$ is of depth (at most) 1, as is $\mathfrak{P}^!$.

Let

$$\overline{\mathfrak{M}} \to \overline{\mathfrak{P}}$$

be the linked space which over a pair $i < 0$ is defined to be

$$\partial_i \xleftarrow{\mathrm{id}} \partial_i \hookrightarrow M^\circ,$$

and

$$\partial_i \xleftarrow{\mathrm{id}} \partial_i \hookrightarrow \partial_i \times \mathbf{1}$$

over $i < i_+$. That is, over $i < 0$ we apply Example 3.2.15 to the manifold-with-boundary $M \setminus \amalg_{j \neq i} \partial_j$, and over $i < i_+$ we apply the same to the manifold-with-boundary $\partial_i \times \mathbf{R}_{\geq 0}$. Further, let

$$\mathfrak{M}^! \to \mathfrak{P}^!$$

be the linked space given over each pair $i < 0$ by

$$\mathbf{0} \xleftarrow{\mathrm{id}} \mathbf{0} \hookrightarrow \mathbf{1},$$

i.e., Example 3.2.15 applied to the half-line.

Finally the map p is given by, first, the poset map

$$\begin{aligned} \overline{\mathfrak{P}} &\to \mathfrak{P}^!, \\ 0, i &\mapsto 0, \\ i_+ &\mapsto i. \end{aligned}$$

Consequently, it is necessarily given by the trivial maps

$$M^\circ, \partial_i \twoheadrightarrow \mathbf{0}$$

on the 0- and i-strata, and again necessarily by $\partial_i \twoheadrightarrow \mathbf{0}$ on the links; and we define it to be the coordinate projections

$$\partial_i \times \mathbf{1} \twoheadrightarrow \mathbf{1}$$

on the i_+-strata.

We call the rule that assigns p to M the P^2 *construction.*[4] We call $\overline{\mathfrak{M}}$ the *collar* of $\mathfrak{M}$, and $\mathfrak{M}^!$ its *dual.*[5]

Remark 7.2.3. The collar $\overline{\mathfrak{M}} \to \overline{\mathfrak{P}}$ is merely the interior M° given a new stratification, i.e., a *refinement* of M°. Namely, we take, for each boundary component ∂_i, the image of the link embedding discussed in Example 3.2.15 as a codimension-1 submanifold inside M°. Since the normal bundle of the boundary is trivial in any bordism, we can present M° as a collar-gluing along

[4]The phrase 'divide and conquer' is attributed, at least by Wikipedia, to Philip II of Macedon, who kindly shared his initial with 'projection'. Fortunately, calling it the 'DAC construction' would have been equally undescriptive.

[5]The idea vaguely resembles that of 'quadratic duals' in [38, §2.1.9]. Longer but more descriptive would have been to call $\overline{\mathfrak{M}}$ 'M divided' and $\mathfrak{M}^!$ 'M conquered.'

this submanifold, i.e., obtain a diffeomorphism

$$M^\circ \cong \partial_i \times \mathbf{R}_{\geq 0} \amalg_{\partial_i} M^\circ.$$

If M has only finitely (say k) many boundary components, we can choose link embeddings with disjoint images, and present M° as a simultaneous k-fold collar-gluing in the same manner, and the linked manifold associated with the resulting stratified manifold is precisely the collar. The purely linked approach accepts arbitrarily many boundary components.

Example 7.2.4. A smooth manifold M (which by our convention means $\partial M = \emptyset$) is assigned by P^2 the trivial map

$$\begin{tikzcd} M \arrow[d, two heads] \\ \mathbf{0} \end{tikzcd} .$$

Example 7.2.5. Let M be a manifold with connected boundary ∂. In light of Remark 7.2.3, the collar as an ordinary stratified space is given by

$$\overline{M} = \partial \times \mathbf{R}_{\leq 0} \coprod_{\partial \times \{0\}} M. \tag{7.2.6}$$

whose associated linked space is

$$\begin{tikzcd} & \partial \arrow[dl, hook', "\mathrm{id}\times\{+\}"'] \arrow[dr, two heads, "="] & & \partial \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_+"] & \\ \partial \times \mathbf{R}_{<0} = \partial \times \mathbf{1} & & \partial & & M^\circ \end{tikzcd} \tag{7.2.7}$$

or

$$\partial \times \mathbf{1} \xhookleftarrow{\mathrm{id}\times\{+\}} \partial \xhookrightarrow{\iota_+} M^\circ \tag{7.2.8}$$

for short (see Figure 3).

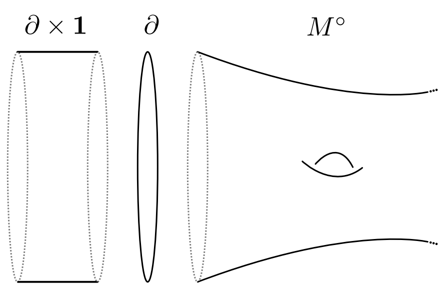


FIGURE 3. The three strata of the linked collar of a manifold with connected boundary.

The dual is then

$$\mathfrak{M}^! = \left(\mathbf{1} \overset{+}{\hookleftarrow} \{0\} = \mathbf{0} \overset{=}{\twoheadrightarrow} \mathbf{0}\right),$$

namely the linked half-line $\mathbf{R}_{\leq 0}$. The projection

$$\begin{tikzcd} \overline{\mathfrak{M}} \arrow[d, two heads, "p"] \\ \mathfrak{M}^! \end{tikzcd}$$

is given by

$$\begin{tikzcd}
& \partial \arrow[dl, hook', "\mathrm{id}\times\{+\}"'] \arrow[dr, two heads, "="] \arrow[dd, dashed, two heads] & & \partial \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_+"] \arrow[dddl, dashed, two heads, bend left] & \\
\partial\times\mathbf{1} \arrow[ddd, two heads, "\mathrm{pr}_2"] & & \partial \arrow[ddd, two heads] & & M^\circ \arrow[dddll, two heads] \\
& \mathbf{0} \arrow[ddl, hook', "+"'] \arrow[ddr, two heads, "="] & & & \\
& & & & \\
\mathbf{1} & & \mathbf{0} & &
\end{tikzcd}, \tag{7.2.9}$$

for which we also write

$$\begin{tikzcd}
\partial\times\mathbf{1} \arrow[d, two heads] & \partial \arrow[l, hook', "\mathrm{id}\times\{+\}"'] \arrow[r, hook, "\iota_+"] \arrow[d, two heads] & M^\circ \arrow[dl, two heads] \\
\mathbf{1} & \mathbf{0} \arrow[l, hook', "+"] &
\end{tikzcd}, \tag{7.2.10}$$

with the above meaning understood.[6]

Example 7.2.11 ($\partial = \partial_L \amalg \partial_R$)**.** Let M be a bordism with $\partial M = \partial_L \amalg \partial_R$ with ∂_L, ∂_R connected. We see M as a linked space $\mathfrak{M}$ of depth 1 but now with three strata:

$$\begin{tikzcd}
& \partial_L \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_{+L}"] & & \partial_R \arrow[dl, hook', "\iota_{+R}"'] \arrow[dr, two heads, "="] & \\
\partial_L & & M^\circ & & \partial_R
\end{tikzcd}.$$

We see that the collar $\overline{\mathfrak{M}}$ in short notation (cf. (7.2.8)), is

$$\partial_L \times \mathbf{1} = \partial_L \times \mathbf{R}_{<0} \xhookleftarrow{\mathrm{id}\times\{+_L\}} \partial_L \xhookrightarrow{\iota_{+L}} M^\circ \xhookleftarrow{\iota_{+R}} \partial_R \xhookrightarrow{\mathrm{id}\times\{+_R\}} \partial_R \times \mathbf{R}_{>0} = \partial_R \times \mathbf{1},$$

the linked version of $\overline{M} = \partial_L \times \mathbf{R}_{\leq 0} \coprod_{\partial_L\times\{0\}} M \coprod_{\partial_R\times\{0\}} \partial_R \times \mathbf{R}_{\geq 0}$, and the dual $\mathfrak{M}^!$ is

$$\begin{tikzcd}
& \mathbf{0} \arrow[dl, hook', "+_L"'] \arrow[dr] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook, "+_R"] & \\
\mathbf{1} & & \mathbf{0} & & \mathbf{1}
\end{tikzcd},$$

the linked version of $M^! = \mathbf{R}_{\{0\}}$, the real line with three-fold stratification given by (say) the defect $\{0\} \subset \mathbf{R}$ and the two components of its complement.

[6]We introduce better notation in Notation 7.2.16 which however obscures the link maps. For this reason, we keep this more verbose version until then.

We write $\mathfrak{M}^! = \left(\mathbf{1} \xhookleftarrow{+_L} \mathbf{0} \xhookrightarrow{+_R} \mathbf{1}\right)$. We observe the projection $p\colon \overline{\mathfrak{M}} \to \mathfrak{M}^!$ to be given, in short notation (cf. (7.2.10)), by

$$\begin{tikzcd}
\partial_L \times \mathbf{1} \arrow[d, two heads] & \partial_L \arrow[l, hook', "\mathrm{id}\times\{+_L\}"'] \arrow[r, hook, "\iota_{+_L}"] & M^\circ \arrow[d, two heads] & \partial_R \arrow[l, hook', "\iota_{+_R}"'] \arrow[r, hook, "\mathrm{id}\times\{+_R\}"] & \partial_R \times \mathbf{1} \arrow[d, two heads] \\
\mathbf{1} & & \mathbf{0} \arrow[ll, hook', "+_L"'] \arrow[rr, hook, "+_R"] & & \mathbf{1}
\end{tikzcd}$$

with the link-wise necessarily trivial.

A distinct advantage of this approach is that it can treat defects on the same footing as boundaries with no modification.

Example 7.2.12 (‘defects as bordisms’: codimension ≥ 2)**.** Let M be a smooth n-manifold and $\Sigma \subset M$ a smooth submanifold of codimension m, which yields a linked manifold $\mathfrak{M}$ with link $\mathbb{S} := L = \mathbb{S}(\mathrm{N}_M\Sigma)$ (see Example 6.1.8). Assume $\mathbb{S}$ is connected, so $m \geq 2$. If $\mathfrak{M}$ is constructible (in that the normal bundle of $\mathbb{S}$ inside $M \smallsetminus \Sigma$ is trivialised), then we can assign to it the collar $\overline{\mathfrak{M}}$, again three-fold stratified, given by

$$\begin{tikzcd}
& \mathbb{S} \arrow[dl, hook'] \arrow[dr, two heads, "="] & & \mathbb{S} \arrow[dl, two heads, "="'] \arrow[dr, hook] & \\
\mathbb{S} \times \mathbf{1} & & \mathbb{S} & & M \smallsetminus \Sigma
\end{tikzcd}$$

and the dual $\mathfrak{M}^!$ can be taken to be the half-line $\left(\mathbf{1} \hookleftarrow \mathbf{0} \overset{=}{\twoheadrightarrow} \mathbf{0}\right)$ together with the obvious projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$. Thus, closed submanifold defects can be treated exactly the same way as bordisms. Analogous considerations apply when one allows boundary components and multiple non-intersecting defect submanifolds.

Remark 7.2.13. Note that the ‘right side’ of the collar in Example 7.2.12, $\mathbb{S} \overset{=}{\twoheadleftarrow} \mathbb{S} \hookrightarrow M \smallsetminus \Sigma$ is simply the linked version of the blow-up (a.k.a. *unzip* in the conically-smooth literature) of M at Σ. In light of the previous examples, we can identify the linked spaces induced by bordisms as those whose blow-ups coincide with themselves. The ‘left side’, in contrast is simply the collar as in the simple codimension-1 case.

Example 7.2.14 (‘defects as bordisms’: codimension 1)**.** If $\Sigma \subset M$ is a smooth submanifold of codimension 1, then its link $L = \mathbb{S}(\mathrm{N}_M\Sigma) \cong \Sigma \amalg \Sigma$ will be two copies of Σ itself. The collar, in short notation, can be chosen to be

$$\overline{\mathfrak{M}} = ((\Sigma \amalg \Sigma) \times \mathbf{1} \hookleftarrow \Sigma \amalg \Sigma \hookrightarrow M \smallsetminus \Sigma)$$

and the dual can be taken to be the half-line again. The projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$ is, exactly as in Example 7.2.12, given by sending the two strata $\Sigma \amalg \Sigma$ and $M \smallsetminus \Sigma$ to $\mathbf{0}$ and the stratum $(\Sigma \amalg \Sigma) \times \mathbf{1}$ to $\mathbf{1}$.

Remark 7.2.15. There is a different approach to closed submanifold defects than Example 7.2.12 which reflects the topology around the defect submanifold better, which is the topic of Section 7.4. The version above should be read as the bare-bones, minimalist approach.

Notation 7.2.16. We will write

$$\begin{tikzcd} M \arrow[r, tail, "L"] & N \end{tikzcd}$$

for a linked space of type $M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N$.

7.3. Corners: a teaser

In this section, we will provide a slightly informal treatment of corners in the linked context, as well as the extension of $\overline{(-)} \twoheadrightarrow (-)^!$ to depth 2. A full theory in arbitrary depth lies beyond the scope of the present work, but we will point to the essential ingredients. One could perhaps also repurpose the idea of generalised links of Douteau–Waas [26, §2.5] for a similar Ansatz. The contents of this section are not needed in any other section, and so it can be skipped.

Figure 4 gives an essentially full picture of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by a manifold M with *a* corner – the passage to multiple corners is analogous to the passage to multiple boundary components (cf. Example 7.2.11).

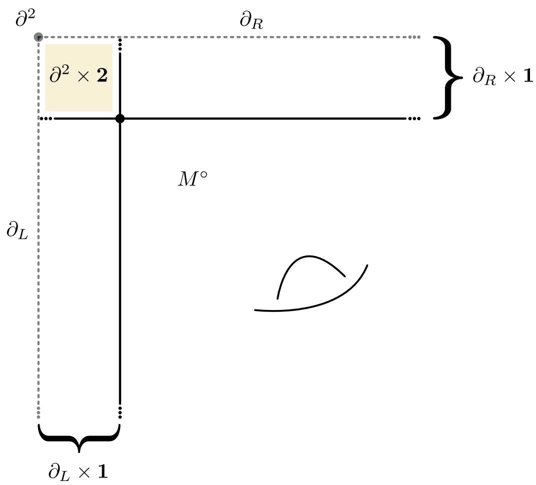


FIGURE 4. Construction of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by M with connected boundary $\partial = \partial_L \cup \partial^2 \cup \partial_R$, with ∂^2 denoting the corner.

Here, $\overline{\mathfrak{M}}$ is a depth-2 linked space with 9 strata: those of full dimension are (isomorphic to) $\partial^2 \times \mathbf{2}$, $\partial_R \times \mathbf{1}$, $\partial_L \times \mathbf{1}$ and M^2; then there is a cross' worth of codimension-1 strata, depicted by solid lines in Figure 4, and finally their codimension-2 intersection, depicted by the solid point.

It is evident from the figure how the depth-2 dual

$$\mathfrak{M}^! = \left(\begin{tikzcd} \partial^2 \arrow[r, tail] \arrow[d, tail] \arrow[dr, tail, dashed] & \partial_R \arrow[d, tail] \\ \partial_L \arrow[r, tail] & M^\circ \end{tikzcd} \right)$$

must be assembled by considering the adjacent depth-1 pairs. This is illustrated in Figure 5.

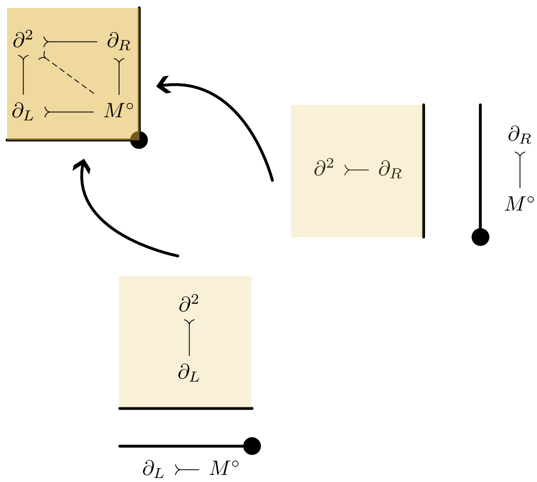


FIGURE 5. The assembly of a corner: $\mathfrak{M}^!$ from depth-1 data.

Now, recall Definition 7.1.1. Indeed, $\mathfrak{M}^!$ is, as a stratified space, the quarter-plane $\mathbf{R}_{\leq 0} \times \mathbf{R}_{\leq 0}$. Let us introduce it as a *product linked space*, the simplest type of linked space of depth 2. Each factor is $\mathbf{0} \overset{\mathbf{0}}{\rightarrowtail} \mathbf{1}$ stratified over $\mathfrak{P} = \{0 < 1\}$, so the product is

$$\begin{tikzcd} \mathbf{0}\times\mathbf{0} \arrow[r, tail, "\mathbf{0}"] \arrow[d, tail, "\mathbf{0}"'] \arrow[dr, tail, "\mathbf{0}\times\mathbf{0}"] & \mathbf{1}\times\mathbf{0} \arrow[d, tail, "\mathbf{0}"] \\ \mathbf{1}\times\mathbf{0} \arrow[r, tail, "\mathbf{0}"'] & \mathbf{1}\times\mathbf{1} \end{tikzcd}$$

stratified over

$$\mathfrak{P} \times \mathfrak{P} = \left\{ \begin{array}{ccc} (0,0) & < & (1,0) \\ \wedge & \rotatebox{45}{$\wedge$} & \wedge \\ (1,0) & < & (1,1) \end{array} \right\}.$$

In a general product of depth-1 linked spaces, the new links will be given, for $(\mathfrak{p}, \mathfrak{q}) \leq (\mathfrak{p}', \mathfrak{q}')$ in $\mathfrak{P} \times \mathfrak{Q}$, by

$$L_{(\mathfrak{p},\mathfrak{q}),(\mathfrak{p}',\mathfrak{q}')} = L_{\mathfrak{p},\mathfrak{p}'} \times L_{\mathfrak{q},\mathfrak{q}'}.$$

In a general linked space of depth 2 (or indeed in any depth) indexed over a poset $\mathfrak{P}$ of depth 2, we ask that, for each concantenation $\mathfrak{p} < \mathfrak{q} < \mathfrak{r}$, commuting diagrams

$$\begin{tikzcd} L_{\mathfrak{pq}} \times_{M_{\mathfrak{q}}} L_{\mathfrak{qr}} \arrow[rr, dotted] \arrow[dr] & & L_{\mathfrak{pr}} \arrow[dl] \\ & M_{\mathfrak{p}} \times M_{\mathfrak{r}} & \end{tikzcd} \tag{7.3.1}$$

be given. We call the maps $L_{\mathfrak{pq}} \times_{M_{\mathfrak{q}}} L_{\mathfrak{qr}} \to L_{\mathfrak{pr}}$ *concatenation* maps and the condition that they cover $M_{\mathfrak{p}} \times M_{\mathfrak{r}}$ that concatenation be *rel endpoints*. The terminology is justified in view Theorem 3.3.1; see also Remark 7.1.12.

Finally, the projection $p\colon \overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ is visible in Figure 4: $\partial_{R/L} \times \mathbf{1}$ project to the two coordinates of the quarter plane, $\partial^2 \times \mathbf{2}$ projects to $\mathbf{2}$, and M° and its closure are collapsed to the corner point of the quarter plane.

7.4. Duals of bordisms with defect submanifolds

In Examples 7.2.12 and 7.2.14 we treated defect submanifolds as if they were boundary components. However, this relies on forgetting some information that one would not wish a defect version of the TFT of a disk algebra to forget.

For instance, a point defect in the plane is assigned the projection

$$\begin{tikzcd} \mathbf{R}^2 \smallsetminus \{0\} \arrow[drr, two heads] & & S^1 \arrow[ll, tail, no head, "S^1"'] \arrow[r, tail, no head, "S^1"] \arrow[d, two heads] & S^1 \times \mathbf{1} \arrow[d, two heads] \\ & & \mathbf{0} \arrow[r, tail, no head] & \mathbf{1} \end{tikzcd}$$

by Example 7.2.12, pushing forward the algebra along which will give a 1-disk algebra with a module. However, one would expect that the point defect, having codimension 2, be assigned a 2-algebra (for an input 2-algebra), acting on the 0-algebra assigned to the bulk. In other words, the construction of Example 7.2.12 integrates out one dimension too many.

We will start with exactly this example.

Example 7.4.1. Let $\mathfrak{M}$ be the euclidean plane with a point defect at the origin. We associate with it the projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$ given as follows:

$$\begin{tikzcd} \mathbf{R}^2 \smallsetminus \{0\} \arrow[drr, two heads] & & S^1 \arrow[ll, tail, no head, "S^1"'] \arrow[r, tail, no head, "S^1"] \arrow[d, two heads] & S^1 \times \mathbf{1} \arrow[d, two heads] \\ & & \mathbf{0} \arrow[r, tail, no head, "S^1"] & S^1 \times \mathbf{1} \end{tikzcd}$$

While this is certainly a valid map of linked spaces, it is not immediately obvious what its ordinary stratified counterpart is.

Let us discuss the restriction to the 'neighbourhood of the defect,' i.e., the square in the diagram above. Realising to stratified spaces in the sense of Definition 7.1.14, we obtain

$$\left| S^1 \overset{S^1}{\rightarrowtail\!\!-\!\!-} S^1 \times \mathbf{1} \right| = \mathbf{D}^2 \smallsetminus \{0\},$$

the closed 2-disk without its origin, equipped with the boundary stratification; and

$$\left| \; \mathbf{0} \overset{S^1}{\rightarrowtail\!\!-\!\!-} S^1 \times \mathbf{1} \; \right| = \mathbf{R}^2_{\{0\}} = |\mathfrak{M}|,$$

the plane with a point defect again. (We will see that all point defects are 'locally self-dual'.) The realised projection

$$|p|\colon \mathbf{D}^2 \smallsetminus \{0\} \to \mathbf{R}^2_{\{0\}} \tag{7.4.2}$$

can be described as the map that sends the whole boundary circle to the origin, and is given by

$$|p|(x) = \left(\frac{1}{|x|} - 1\right) x$$

away from the boundary.

In the framed case, the input $\mathbb{E}_2$-algebra A induces a factorisation algebra F_A on the bulk $S^1 \times \mathbf{1} \cong \mathbf{R}^2 \smallsetminus \{0\}$ of the target space (in the usual way: see [36, §4.2]; also [37]), and a constructible factorisation algebra on the latter is equivalent to the datum of an $\mathbb{E}_1$-module over $\int_{S^1\times\mathbf{R}} A$. Indeed, $\int_{S^1\times\mathbf{R}} A \simeq U_A$ is the universal enveloping $\mathbb{E}_1$-algebra of A, either by definition as in [36, §7.1], or as a theorem as in [33, Proposition 3.16], with a more algebraic definition in the spirit of [38, §1.6] given in [33, Definition 2.5].

We generalise the idea of Example 7.4.1 in the following definition. For simplicity, we will consider a single closed defect submanifold and leave the details of a treatment of an arbitrary number of non-intersecting ones to the reader, which is analogous to Definition 7.2.2.

Definition 7.4.3 (P^2 on defects)**.** Let $\Sigma \subset M$ be a closed smooth submanifold of codimension k with trivialised normal bundle $\mathrm{N} = \mathrm{N}_M\Sigma$. The P^2 *construction on defects* assigns to the linked space $\mathfrak{M}$ associated with the stratified space $\Sigma \subset M$ the projection $p\colon \overline{\mathfrak{M}} \to \mathfrak{M}^!$ given by

$$\begin{tikzcd}
M \smallsetminus \Sigma \arrow[dr, two heads] & \Sigma \times S^{k-1} \arrow[l, tail, "\Sigma\times S^{k-1}"'] \arrow[r, tail, "\Sigma\times S^{k-1}"] \arrow[d, two heads] & \Sigma \times S^{k-1} \times \mathbf{1} \arrow[d, two heads] \\
& \mathbf{0} \arrow[r, tail, "S^{k-1}"'] & S^{k-1} \times \mathbf{1}
\end{tikzcd} \qquad (7.4.4)$$

Example 7.4.5. If $\Sigma \subset M$ has codimension 1, then

$$\left|\mathfrak{M}^!\right| = \mathbf{R}_{\{0\}},$$

a line with a point defect, but with only two strata, and so with link $\{\pm\}$.

Remark 7.4.6. Similarly to the case of a bordism (cf. Remark 7.2.3), the collar is the 'interior' with a new stratification, in that $\overline{\mathfrak{M}}$ is the linked space associated with

$$M \smallsetminus \Sigma \cong (M \smallsetminus \Sigma) \amalg_{\Sigma\times S^{k-1}} \Sigma \times S^{k-1} \times \mathbf{R}_{\geq 0}.$$

The square in (7.4.4), the 'neighbourhood of the defect,' is given after linked realisation by the boundary-collapsing, norm-reversing map

$$|p|\colon \mathbb{D}(\mathrm{N}) \smallsetminus \Sigma \to \mathbf{R}^k_{\{0\}}$$

generalising Example 7.4.1. Here, the domain is the disk bundle of the normal bundle of Σ with the zero section taken out, equipped with the boundary stratification. The map $|p|$ sends the boundary $\mathbb{S}(\mathrm{N})$ to the defect $\{0\}$, and is otherwise given, using a trivialisation of the normal bundle and a metric, by

$$\begin{aligned} (\mathbb{D}(\mathrm{N}) \smallsetminus \Sigma)^\circ \cong \mathbb{S}(\mathrm{N}) \times \mathbf{R} & \\ & \cong \Sigma \times S^{k-1} \times (0,1) \to \mathbf{R}^k \smallsetminus \{0\}, \\ & (p,q,x) \mapsto \left(\frac{1}{|x|} - 1\right) q. \end{aligned}$$

Finally, we will combine Definitions 7.2.2 and 7.4.3, and again, for simplicity, will consider a single boundary component and a single defect submanifold, leaving the repetition for mutliple such to the reader; it is also spelled out in the proof of Proposition 7.5.5.

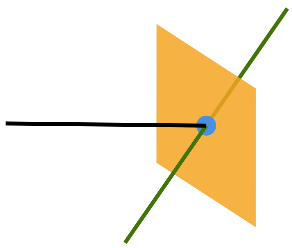
Definition 7.4.7 (P^2 on bordisms with defects)**.** Let $\Sigma \subset M^\circ$ be a smooth closed submanifold of codimension k with trivialised normal bundle in the interior of a manifold M with connected boundary ∂. We define its *bulk* to be

$$M^\circ := M \smallsetminus \{\Sigma \amalg \partial\}.$$

The P^2 construction assigns to M stratified by Σ and ∂ the projection

$$\begin{tikzcd} \partial \times \mathbf{1} \arrow[r, no head, tail reversed, "\partial"] \arrow[dr, two heads] & \partial \arrow[r, no head, tail, "\partial"] \arrow[dr, two heads] & M^\circ \arrow[d, two heads] & \Sigma \times S^{k-1} \arrow[l, tail, no head, "\Sigma\times S^{k-1}"'] \arrow[dl, two heads] \arrow[r, no head, tail, "\Sigma\times S^{k-1}"] & \Sigma \times S^{k-1} \times \mathbf{1} \arrow[dl, two heads] \\ & \mathbf{1} \arrow[r, no head, tail reversed, "\mathbf{0}"'] & \mathbf{0} \arrow[r, no head, tail, "S^{k-1}"'] & S^{k-1} \times \mathbf{1} & \end{tikzcd}.$$

Example 7.4.8. If $\mathfrak{M}$ is (the linked space associated with) a bordism with a single boundary component and two defect submanifolds with trivialised normal bundle, one of codimension 2 and one of codimension 1, then $\left|\mathfrak{M}^!\right|$ is



The middle ball is the unique point-stratum $\mathbf{0}$, the black half-line depicts the boundary component conquered,[7] the green half-lines depict the codimension-1 defect conquered and make up a single stratum (see Example 7.4.5), and the orange plane through $\mathbf{0}$ depicts the codimension-2 defect conquered.

Remark 7.4.9. The realisations $|\mathfrak{M}^!|$ of the conquered manifolds are conically smooth: for instance, the real plane with a half-line attached to it (as in part of the picture in Example 7.4.8) is the stratified open cone $C(* \amalg S^1)$, a conically-smooth basic. The general case is analogous. Similarly, the realised collar $|\overline{\mathfrak{M}}|$ is conically-smooth, being again simply the bulk $M \smallsetminus \{\coprod \Sigma_i \amalg \coprod \partial_j\}$ with a new stratification by mutually disjoint closed submanifolds.

[7]See Footnote 5.

7.5. Cutting and gluing

We will suspend Notation 7.2.16 briefly for the sake of clarity – it will come back into effect momentarily.

Definition 7.5.1. Let M be a bordism, with or without defect submanifolds. A *cut locus* $\iota\colon \Sigma \subset M^\circ$ is a closed codimension-1 submanifold in the interior in the sense of Definition 7.4.7, together with a diffeomorphism $M \cong M_L \cup_{\Sigma\times\mathbf{R}} M_R$, a *collar-gluing* of M (or of bordisms M_L and M_R) along Σ. These data are organised as a linked space $\mathfrak{M}_\Sigma$ in the obvious manner by mixing the examples in Sections 7.2 and 7.4.

We will disregard the choice of diffeomorphism in the collar-gluing, and, for simplicity, assume M has only two boundary components, ∂_L and ∂_R, and no defect submanifolds. We will include these back in within the proof of Proposition 7.5.5

Definition 7.5.2. By *cutting-and-gluing* (*of* M *along* Σ) we mean the *refinement* map

$$r = r_\Sigma\colon \mathfrak{M}_\Sigma \to \mathfrak{M},$$

one which is stratum-wise a diffeomorphism,[8] given, after decorating the outward copies of the cut locus with ℓ and r, by

$$\begin{tikzcd}
\partial_L \arrow[d] & {\color{blue}\partial_L} \arrow[l, two heads] \arrow[r, hook] \arrow[drr] & M_L^\circ \arrow[drrrr] & {\color{blue}\Sigma_\ell} \arrow[l, hook'] \arrow[r, two heads] & \Sigma_\ell \arrow[drr] & {\color{blue}\Sigma_\ell} \arrow[l, two heads] \arrow[r, hook] & \Sigma\times\mathbf{1} \arrow[d] & \cdots \\
\partial_L & & & {\color{blue}\partial_L} \arrow[lll, two heads] \arrow[rrr, hook] & & & M^\circ &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Sigma\times\mathbf{1} \arrow[d] & {\color{blue}\Sigma_r} \arrow[l, hook'] \arrow[r, two heads] & \Sigma_r \arrow[dll] & {\color{blue}\Sigma_r} \arrow[l, two heads] \arrow[r, hook] & M_R^\circ \arrow[dllll] & {\color{blue}\partial_R} \arrow[l, hook'] \arrow[r, two heads] \arrow[dll] & \partial_R \arrow[d] \\
\cdots & M^\circ & & & {\color{blue}\partial_R} \arrow[lll, hook'] \arrow[rrr, two heads] & & & \partial_R
\end{tikzcd}$$

where the vertical maps are all inclusions and identities. For better legibility and distinguishability, the links are coloured blue.

The following definition extends P^2 to situations where M comes with a specified collar gluing. It is slightly redundant (∂M has two components) as well as incomplete (since defects – and different numbers of boundary components – are missing), but we give the obvious extension in the proof of Proposition 7.5.5

Definition 7.5.3 (The P^2 construction with a cut locus)**.** We resume working in the situation above. Writing $M_L^\circ = M_L \smallsetminus \partial_L$ and $M_R^\circ = M_R \smallsetminus \partial_R$, we define

[8]This is a barbarian notion of recollement that will suffice for our purposes.

the *cut collar* $\overline{\mathfrak{M}_\Sigma}$ to be

$$\begin{tikzcd}[column sep=small]
& \partial_L \arrow[dl, hook] \arrow[dr, two heads] & & \partial_L \arrow[dl, two heads] \arrow[dr, hook] & & \Sigma \arrow[dl, hook] \arrow[dr, two heads] & & \Sigma \arrow[dl, two heads] \arrow[dr, hook] & & \cdots \\
\partial_L \times \mathbf{1} & & \partial_L & & M_L^\circ & & \Sigma & & \Sigma \times \mathbf{1} &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots & & \Sigma \arrow[dl, hook] \arrow[dr, two heads] & & \Sigma \arrow[dl, two heads] \arrow[dr, hook] & & \partial_R \arrow[dl, hook] \arrow[dr, two heads] & & \partial_R \arrow[dl, two heads] \arrow[dr, hook] & \\
& \Sigma \times \mathbf{1} & & \Sigma & & M_R^\circ & & \partial_R & & \partial_R \times \mathbf{1}
\end{tikzcd}$$

the linked version of

$$\overline{M_\Sigma} = \partial_L \times \mathbf{R}_{\leq 0} \cup_{\partial_L \times \{0\}} M_L \cup_{\Sigma \times \mathbf{R}_{<1}} \Sigma \times \mathbf{R} \cup_{\Sigma \times \mathbf{R}_{>-1}} M_R \cup_{\partial_R \times \{0\}} \partial_R \times \mathbf{R}_{\geq 0}.$$

The *cut dual* $\mathfrak{M}^!_\Sigma$ is

$$\begin{tikzcd}[column sep=small]
& \mathbf{0} \arrow[dl, hook] \arrow[dr, two heads] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook] & & \mathbf{0} \arrow[dl, hook] \arrow[dr, two heads] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook] & \\
\mathbf{1} & & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & & \mathbf{1}
\end{tikzcd},$$

the linked version of $\mathbf{R}_{\{\pm 1\}}$, the real line with defects $\{-1\}$, $\{1\}$. The projection $p_\Sigma \colon \overline{\mathfrak{M}_\Sigma} \to \mathfrak{M}^!_\Sigma$ is, using Notation 7.2.16, as follows:

$$\begin{tikzcd}[column sep=small]
\partial_L \times \mathbf{1} \arrow[d, two heads] & \partial_L \arrow[l, tail] \arrow[r, tail] \arrow[dr, two heads] & M_L^\circ \arrow[d, two heads] & \Sigma \arrow[l, tail] \arrow[r, tail] \arrow[dl, two heads] & \Sigma \times \mathbf{1} \arrow[d, two heads] & \Sigma \arrow[l, tail] \arrow[r, tail] \arrow[dr, two heads] & M_R^\circ \arrow[d, two heads] & \partial_R \arrow[l, tail] \arrow[r, tail] \arrow[dl, two heads] & \partial_R \times \mathbf{1} \arrow[d, two heads] \\
\mathbf{1} & & \mathbf{0} \arrow[ll, tail] \arrow[rr, tail] & & \mathbf{1} & & \mathbf{0} \arrow[ll, tail] \arrow[rr, tail] & & \mathbf{1}
\end{tikzcd}$$

Remark 7.5.4. In order to conform to Definition 7.1.2, it remains to specify

$$\begin{tikzcd}
\Sigma_{\ell/r} \arrow[r, dotted] \arrow[d] & (M^\circ)^I \arrow[d] \\
\Sigma_{\ell/r} \times M^\circ_{L/R} \arrow[r] & M^\circ \times M^\circ
\end{tikzcd}, \qquad
\begin{tikzcd}
\Sigma_{\ell/r} \arrow[r, dotted] \arrow[d] & (M^\circ)^I \arrow[d] \\
\Sigma_{\ell/r} \times (\Sigma \times \mathbf{1}) \arrow[r] & M^\circ \times M^\circ
\end{tikzcd},$$

which are given analogously to one another. For instance, for the map $\Sigma_\ell \to (M^\circ)^I$ on the left, first reparametrise the gluing line as $\mathbf{R} \cong (-1, 1)$ such that $\Sigma \times \mathbf{1} = \Sigma \times (-1, 1)$ is the middle embedding. Extend now the tubular neighbourhood of Σ to times $(-2, 2) \supset (-1, 1)$. The link embedding $\Sigma_\ell \hookrightarrow M_L^\circ$ is given by following for a nonzero time τ with $-1 > \tau > -2$ the outward nonvanishing vector field X_Σ along Σ that frames the rank-1 normal bundle of $\Sigma \subset M$ (whereas $\Sigma_r \hookrightarrow M_R^\circ$ is determined by following it for time $-\tau$), while the stratum map $\Sigma_\ell \to M^\circ$ hits the copy at time $-1 \in (-2, 2)$. Now, the map $\Sigma_\ell \to (M^\circ)^I$ at $p \in \Sigma_\ell$ can be chosen to be the path from p at time -1 to p at time τ along X_Σ. This is in essence the construction discussed in Remark 7.1.12.

The following is a completely straightforward statement that formalises the sense in which P^2 is compatible with cutting and gluing, and its proof consists purely in bookkeeping.

Proposition 7.5.5. *The P^2 construction is compatible with cutting and gluing. That is, it is covariant along refinements, in that the refinement $r\colon \mathfrak{M}_\Sigma \to \mathfrak{M}$ induces maps $\bar{r}\colon \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ and $r^!\colon \mathfrak{M}^!_\Sigma \to \mathfrak{M}^!$ such that*

$$\begin{tikzcd} \overline{\mathfrak{M}_\Sigma} \arrow[r, "\bar{r}"] \arrow[d, two heads, "p_\Sigma"'] & \overline{\mathfrak{M}} \arrow[d, two heads, "p"] \\ \mathfrak{M}^!_\Sigma \arrow[r, "r^!"'] & \mathfrak{M}^! \end{tikzcd} \tag{7.5.6}$$

commutes.

Proof. Let $M = M_L \cup_{\Sigma\times\mathbf{R}} M_R$ be a collar-gluing of a bordism M with defect submanifolds $\Sigma_i^L \subset M_L$, $\Sigma_i^R \subset M_R$ of codimensions $k_{i,L}$ resp. $k_{i,R}$ with trivialised normal bundles, and boundary components $\partial_j^L \subset M_L$, $\partial_j^R \subset M_R$. We will first set up the poset square

$$\begin{tikzcd} \overline{\mathfrak{P}_\Sigma} \arrow[r, "\bar{\mathfrak{r}}"] \arrow[d, "\mathfrak{p}_\Sigma"] & \overline{\mathfrak{P}} \arrow[d, "\mathfrak{p}"] \\ \mathfrak{P}^!_\Sigma \arrow[r, "\mathfrak{r}^!"] & \mathfrak{P}^! \end{tikzcd}$$

over which (7.5.6) is defined. The featured posets were defined mostly implicitly, so we will explicate them.

The poset $\mathfrak{P}$ of $\mathfrak{M}$ itself is generated by $\{0, i^{L/R}_{(+)}, j^{L/R}_{(+)}\}_{i,j}$ and arrows $i^{L/R}, j^{L/R} < 0$ and $i^{L/R} < i^{L/R}_+$, $j^{L/R} < j^{L/R}_+$. The poset $\overline{\mathfrak{P}_\Sigma}$ is generated by

$$\{\mathbf{L}, c_L, \mathbf{C}, c_R, \mathbf{R}, i^{L/R}_{(+)}, j^{L/R}_{(+)}\}_{i,j}$$

with arrows

$$\begin{gathered} i/j^{L/R} < i/j^{L/R}_+, \\ i^{L/R}, j^{L/R} < \mathbf{L}/\mathbf{R}, \\ c_{L/R} < \mathbf{C}, \\ c_{L/R} < \mathbf{L}/\mathbf{R}. \end{gathered}$$

The cut collar is over the latter poset in that the $\mathbf{L}/\mathbf{R}$-strata are $M^\circ_{L/R}$, the $c_{L/R}$-strata are $\Sigma_{\ell/r}$, the $\mathbf{C}$-stratum is $\Sigma \times \mathbf{1}$, and the $i/j^{L/R}$- and $j^{L/R}$-strata are $\Sigma_i^{L/R} \times S^{k_{i,L/R}-1}$ resp. $\partial_j^{L/R}$, and finally the $i_+^{L/R}$- and $j_+^{L/R}$-strata are $\Sigma_i^{L/R} \times S^{k_{i,L/R}-1} \times \mathbf{1}$ resp. $\partial_j^{L/R} \times \mathbf{1}$.

The elements in the poset square are mapped as follows:

$$\begin{tikzcd} \mathbf{L}, c_L, \mathbf{C}, c_R, \mathbf{R}, i^{L/R}_{(+)}, j^{L/R}_{(+)} \arrow[r, mapsto, "\bar{\mathfrak{r}}"] \arrow[d, mapsto, "\mathfrak{p}_\Sigma"] & 0, i^{L/R}_{(+)}, j^{L/R}_{(+)} \arrow[d, mapsto, "\mathfrak{p}"] \\ c_L, \mathbf{C}, c_R, i^{L/R}, j^{L/R} \arrow[r, mapsto, "\mathfrak{r}^!"] & 0, i^{L/R}, j^{L/R} \end{tikzcd}$$

Here,

$$\mathfrak{p}_\Sigma(\mathbf{L}/\mathbf{R}) = \mathfrak{p}_\Sigma(i/j^{L/R}) = c_L/c_R$$

and

$$\mathfrak{p}_\Sigma(i/j_+^{L/R}) = i/j^{L/R}$$

and similarly for $\mathfrak{p}$. The vertical maps send $i/j_+^{L/R} \mapsto i/j^{L/R}$. In $\mathfrak{P}^!_\Sigma$ we have $c_L < i^{L/R}$. The upper horizontal map is

$$\begin{gathered}\bar{\mathfrak{r}}\colon \overline{\mathfrak{P}_\Sigma} \to \overline{P},\\ \mathbf{L}, \mathbf{R}, \mathbf{C}, c_{L/R} \mapsto 0\end{gathered}$$

and otherwise the identity, while the lower horizontal map is

$$\begin{gathered}\mathfrak{r}^!\colon \mathfrak{P}^!_\Sigma \to \mathfrak{P}^!\\ c_L, \mathbf{C}, c_R \mapsto 0\end{gathered}$$

and otherwise also the identity.

The poset square thus constructed clearly commutes, and the span maps constituting (7.5.6) over this poset square can be given by the obvious projections. Evidently, it also commutes. $\square$

For M as before, i.e., with only two boundary components and no defect submanifolds, $\overline{r}\colon \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ is

$$\begin{tikzcd}
\partial_L \times \mathbf{1} \arrow[d] & \partial_L \arrow[l, -<] \arrow[r, >-] \arrow[d] & M_L^\circ \arrow[drr] & \Sigma \arrow[l, -<] \arrow[r, >-] \arrow[dr] & \Sigma \times \mathbf{1} \arrow[d] & \Sigma \arrow[l, -<] \arrow[r, >-] \arrow[dl] & M_R^\circ \arrow[dll] & \partial_R \arrow[l, -<] \arrow[r, >-] \arrow[d] & \partial_R \times \mathbf{1} \arrow[d] \\
\partial_L \times \mathbf{1} & \partial_L \arrow[l, -<] \arrow[rrr, >-] & & & M^\circ & & & \partial_R \arrow[lll, -<] \arrow[r, >-] & \partial_R \times \mathbf{1}
\end{tikzcd}$$

using Notation 7.2.16, and $r^!\colon \mathfrak{M}^!_\Sigma \to \mathfrak{M}^!$ is

$$\begin{tikzcd}
\mathbf{1} \arrow[d] & \mathbf{0} \arrow[l, -<] \arrow[r, >-] \arrow[dr] & \mathbf{1} \arrow[d] & \mathbf{0} \arrow[l, -<] \arrow[r, >-] \arrow[dl] & \mathbf{1} \arrow[d] \\
\mathbf{1} & & \mathbf{0} \arrow[ll, -<] \arrow[rr, >-] & & \mathbf{1}
\end{tikzcd},$$

the linked version of the map $\mathbf{R}_{\{\pm 1\}} \twoheadrightarrow \mathbf{R}_{\{0\}}$ that collapses $[-1, 1]$ onto $\{0\}$ and scales up the two sides of the former to $\mathbf{R}_{<0}$ and $\mathbf{R}_{>0}$ (much like in [67]). We see by direct inspection that (7.5.6) commutes. See Figure 6.

Example 7.5.7. For M_L as in Example 7.4.8 and M_R with a single boundary component a single codimension-1 defect with trivialised normal bundle, $\left|\mathfrak{M}^!_\Sigma\right|$

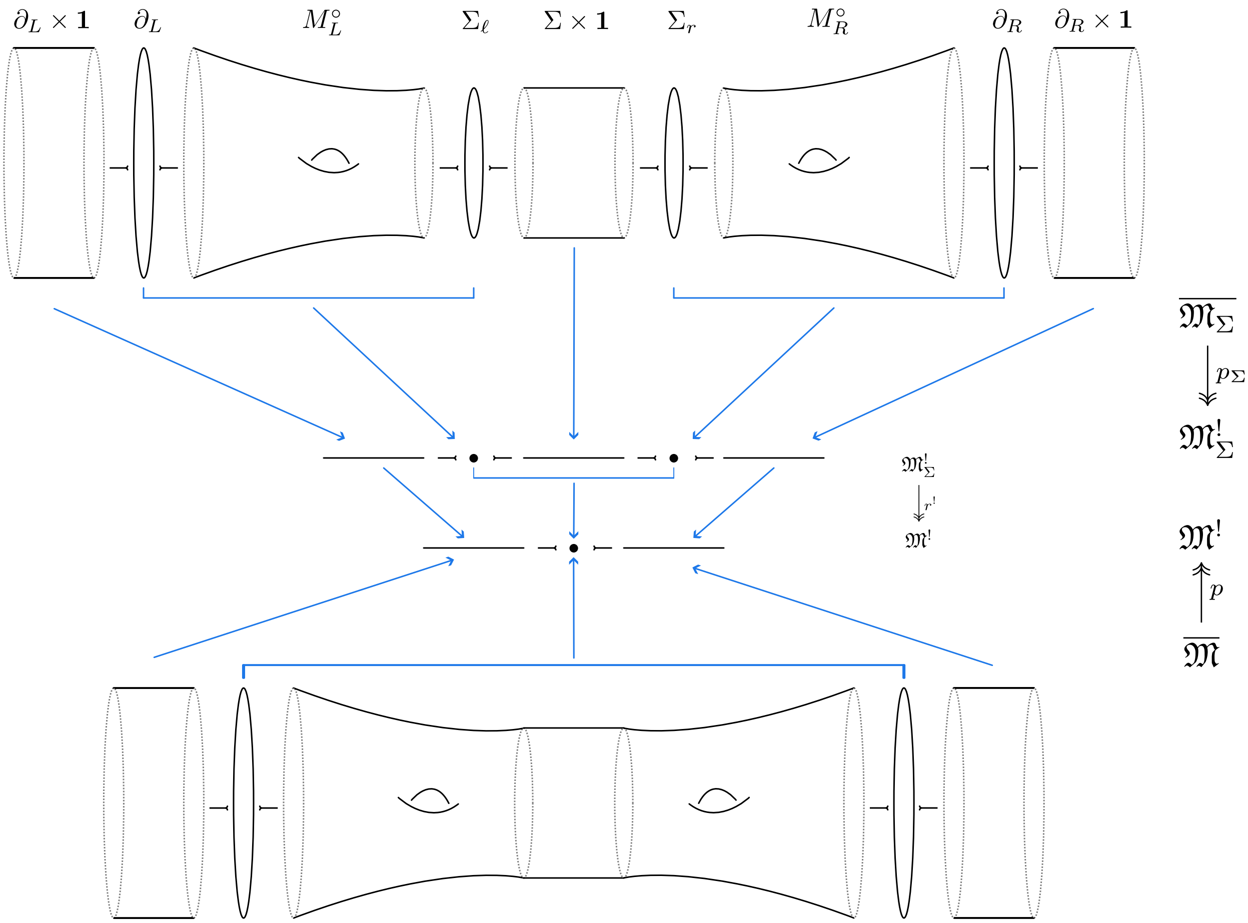
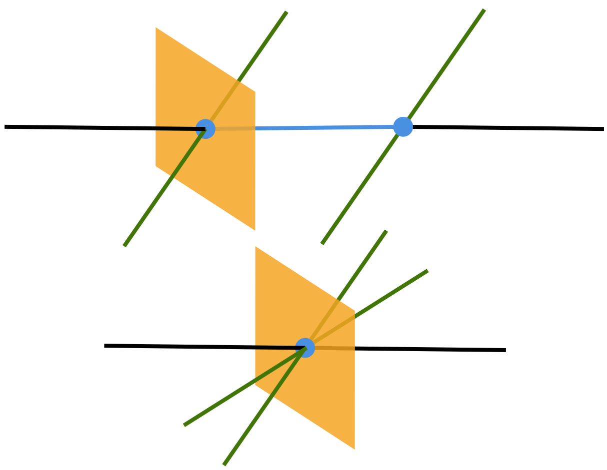


FIGURE 6. The cutting-and-gluing square (7.5.6) using Notation 7.2.16; $\overline{r}\colon \overline{\mathfrak{M}}_\Sigma \to \overline{\mathfrak{M}}$ omitted.

and $\left|\mathfrak{M}^!\right|$ look as follows:



The map $r^!\colon \mathfrak{M}^!_\Sigma \to \mathfrak{M}^!$ collapses the blue closed interval and is otherwise the identity.

Remark 7.5.8. In our discussion of P^2 we have treated connected components separately. However, multiple boundary components and/or defect submanifolds can be grouped together and then fed into P^2, and the resulting dual will have fewer strata as well. Grouping boundary components is straightforward, and the dual will have single half-line per group, reproducing, in the absence of defects, pictures from [67] upon linked realisation. In particular, when one has a single group ('no orientation') or two groups (one 'incoming' and one 'outgoing'), the realised duals, being a half-line or a line with a point defect respectively, will again be conically-smooth.

There is no obvious way of grouping defects of differing codimensions, however, except for simply extending the corresponding links and higher strata in the dual with disjoint unions. If Σ_1 and Σ_2 have the same codimension $k = k_1 = k_2$, then they can be grouped as if they are boundary components since $\Sigma_1 \times S^{k_1-1} \amalg \Sigma_2 \times S^{k_2-1} = (\Sigma_1 \amalg \Sigma_2) \times S^{k-1}$, but otherwise not. Mixing boundary components and defects is ill-advised. Of course, it all depends on what sort of labelling system one wishes to introduce on bordisms with defects.

7.6. P^2 with tangential structure

A choice of ambient smooth tangential structure can be incorporated in a straightforward way, but in a somewhat restricted case, into the P^2 construction to obtain a structure-sensitive version of it, as we will now briefly explain.

Let us fix a natural number $R \geq 0$, a smooth tangential structure $F\colon Y \to BO(R) \hookrightarrow \mathcal{V}^{\hookrightarrow}$, and a bordism M with mutually disjoint closed defect submanifolds in its interior, with $\mathfrak{M}$ denoting the associated linked manifold.

For simplicity, assume $\partial = \partial M$ is connected and that there is a single such submanifold $\Sigma \subset M \smallsetminus \partial$. We will resume the notation $M^\circ = M \smallsetminus \{\partial \amalg \Sigma\}$ in this context, and write $n + m = \dim(M^\circ)$.

Definition 7.6.1. Let $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked manifold with $\dim(M) = n$, $\dim(N) = n+m = R$, for simplicity over the poset [1] as depicted. A *stable Y-structure* on $\mathfrak{S}$ is a solid Y-structure as in Definition 6.4.18 where the solid structures on M and N are stable in the sense of Definition 6.3.6.

If $R > n + m$, then a *stable Y-structure* on $\mathfrak{S}$ is a stable Y-structure on $\mathfrak{S} \times \mathbf{R}^{R-n-m}$.

Remark 7.6.2. When $R = n + m$, a stable structure on $\mathfrak{S}$ according to Definition 7.6.1 is equivalent, by Theorem 6.4.20, to a cartesian structure with stratum-wise stable structures. One could refine this definition for different goals than ours, and we leave this to the interested reader. When $R > n+m$, the definition is already much more restrictive.

Lemma 7.6.3. *Let $\mathfrak{S} = \mathfrak{M}$ be associated with a manifold M with a closed submanifold defect $\Sigma \subset M$ of codimension k, and suppose it has a stable Y-structure. Then $L = \mathbb{S}(\mathrm{N}\Sigma)$ is a trivial sphere bundle over Σ.*

Proof. First, assume $R = n + m = n + k$. In the stable case, the isomorphism $\pi^*W \cong \mathrm{N}_N M$ of bundles over L from Definition 6.4.18 reads $\varepsilon^k \cong (\pi^*\mathrm{T}\Sigma)^\perp \subset \iota^*\mathrm{T}M$ over $L = \mathbb{S}(\mathrm{N}\Sigma)$. We can replace $\iota^*\mathrm{T}M$ by $\iota^*\mathrm{T}(\mathrm{N}\Sigma)$ since, by $\iota\colon \mathbb{S} \hookrightarrow M$ factors through $\mathrm{N}\Sigma$. Let us also write π for the projection $\mathrm{N}\Sigma \to \Sigma$. We have $\mathrm{TN}(\Sigma) \cong \pi^*(\mathrm{T}\Sigma) \oplus \pi^*(\mathrm{N}\Sigma)$, and along the diffeomorphism $\mathrm{N}_0 \coloneqq \mathrm{N}(\Sigma) \smallsetminus \Sigma \cong \mathbb{S} \times \mathbf{R}$, the stable structure pulls back and gives $\pi^*(\mathrm{N}\Sigma) \cong \varepsilon^k$ over N_0. We thus have $\mathrm{T}(\mathrm{N}_0) \cong \pi^*(\mathrm{T}\Sigma) \oplus \varepsilon^k$, yielding $\mathrm{N}_0 \cong \Sigma \times (\mathbf{R}^k \smallsetminus \{0\})$. If $R > n + m$, we similarly obtain $\mathrm{T}(\mathrm{N}_0) \cong \pi^*(\mathrm{T}\Sigma) \oplus \varepsilon^{R-n}$ and therefore the statement. $\square$

Lemma 7.6.3 implies that we have all we need to carry out the P^2 construction:

Definition 7.6.4 (P^2 with stable smooth structures)**.** Let M be a bordism with defects, with bulk dimension $n + m$, and a stable Y-structure. The P^2 *construction* on M is defined to be the same as Definition 7.4.7 except that all spaces and maps involved therein are to be multiplied by the trivial factor $\mathbf{R}^{R-(n+m)}$.

The examples in the previous chapters apply with the modification indicated in Definition 7.6.4.

Example 7.6.5. Suppose M has a single boundary component ∂, so $\mathfrak{M} = (\partial \leftarrow \partial \hookrightarrow M^\circ)$. Say $\dim(M^\circ) = n$. A stable n-framing on $\mathfrak{M}$ is a framing on M° and a framing on $\partial \times \mathbf{1}$ such that the framing on M° pulled back to $\partial \times \mathbf{1}$ and the former are equivalent, in the sense that there is a compatibility map relating them.

Example 7.6.6. The simplest non-trivial example where Definition 7.6.4 does *not* apply is of course when M is the trivially-stratified open Möbius strip and the smooth tangential structure in question is rank-3 framings (3-*framings* for short). The standard embedding of M into $\mathbf{R}^3$ induces a cartesian 3-framing which cannot be promoted to a stable 3-framing since M is not orientable, implying that the normal bundle to the embedding is non-trivial.

7.7. The cartesian problem

Remark 7.7.1. The examples above show clearly that $\mathfrak{M}^!$ parametrises the normal bundle of the stable Y-structure on $\mathfrak{M}$. In non-trivially stratified cases such as Example 7.2.5, however, we see that the classifying map of the normal bundle reverses stratification, which is why we only speak explicitly about the stratification-preserving projections $\overline{\mathfrak{M}} \to \mathfrak{M}^!$. That oughtn't suggest that it is impossible to express classifiers of stratified normal bundles within an appropriate framework. On the other hand, when one starts considering non-smooth tangential structures, the notions of collar and quadratic dual become very problematic because the rank of the normal bundle may both increase or decrase as one moves along strata, since there is no fixed top rank. This is true, for instance, for essentially all variframed spaces (see Example 6.2.5). This makes a general description of $\mathfrak{M}^!$ very hard, if not impossible. This is the main reason we have restricted ourselves to (relaxations of) smooth tangential structures.

We will conclude this chapter with a brief discussion on trying to extend P^2 to deal with examples such as Example 7.6.6.

It is not clear how best to generalise the P^2 construction to the case where $\mathfrak{M}$ only has a cartesian Y-structure. More specifically, we can give a good definition for the collar $\overline{\mathfrak{M}}$, but there is no immediate candidate for $\mathfrak{M}^!$, or for p, for related reasons. We will only present a rather trivial extension that works for the simple fact that we always have a diffeomorphism $\mathrm{N}\Sigma \smallsetminus \Sigma \cong \mathbb{S}(\mathrm{N}\Sigma) \times \mathbf{R}$ using the metric.

To illustrate, suppose first that M is smooth (trivially-stratified) and possesses a not necessarily stable cartesian Y-structure specified by a bundle embedding $\mathrm{T}M \hookrightarrow F^M$, and suppose $R - N \geq 1$. The total space E of the normal bundle $\pi\colon E \to M$ to this embedding carries an induced Y-structure, since $\mathrm{T}E \cong \pi^*(\mathrm{T}M) \oplus \pi^*(E)$, the isomorphism being specified by the metric.[9] This provides a recipe for evaluating a(n unstratified) disk algebra A with Y-structure on M given its cartesian Y-structure t: M provides the space E with Y-structure, which can (be) evaluate(d by) the disk algebra:

$$\mathcal{Z}_A\colon (M,t) \mapsto \int_E A.$$

However, the result is merely a pointed object, but M has codimension $R-N \geq 1$ with respect to the top rank, and should therefore be assigned an $(R-N)$-dimensional algebraic object. The way the stable case solves this problem is by using the 'dual projection' $M \times \mathbf{R}^{R-N} \to \mathbf{R}^{R-N}$ from the bundle E, and pushing A forward along it. There is no such obvious dual projection in general.

The obvious extension of P^2 one can still provide in such a situation simply assigns the projection $\mathbb{S}(E) \times \mathbf{R} \to \mathbf{R}$, which extracts a 1-dimensional algebra out of (M,t) and A. There is an obvious extension to the case where M is a bordism with defects that otherwise mimics the previous P^2 construction of Definition 7.6.4.

This is clearly unsatisfactory if $R-N \geq 2$, and even if $R-N = 1$ as in the cartesian 3-framed open Möbius strip of Example 7.6.6. This highlights the need for a twisted version of P^2 that reflects the twistedness of E. We leave this to future work.

Remark 7.7.2. One *can* apply the P^2 construction to bordisms M *without* defects even if they only have a cartesian Y-structure, since the normal bundle along a boundary component is always trivialisable. If $\dim(M) = n < R$, then we can ask that $M \times \mathbf{R}^{R-n}$ – thus retaining a stable flavour – have a cartesian Y-structure, and proceed as in Definition 7.4.7. Since the collar is always a

[9]Cf. the proof of Lemma 7.6.3. The bundle π^*E is canonically isomorphic to the kernel of the differential of π; the metric merely splits the ensuing short exact sequence of bundles over E. We say 'the' metric, but if the reader wishes to let it remain a variable, the choice can be absorbed into the structure on 'M'.

refinement of M° (or of $M^\circ \times \mathbf{R}^{R-n}$), the cartesian structure still induces a Y-structure on M° (or on $M^\circ \times \mathbf{R}^{R-n}$), and so the map $p\colon \overline{M} \to M^!$ is again a constructible bundle along which we can push forward a Y-structured disk algebra.

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